UNSTABLE SPLITTINGS OF CLASSIFYING SPACES OF p–COMPACT GROUPS

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Abstract. Dwyer and Wilkerson gave a definition of a p–compact group, which is a loop space with certain properties and a good generalization of the notion of compact Lie groups in terms of classifying spaces and homotopy theory; e.g. every p–compact group has a maximal torus, a normalizer of the maximal torus and a Weyl group. The belief or hope that p–compact groups enjoy most properties of compact Lie groups establishes a program for the classification of these objects. Following the classification of compact connected Lie groups, one step in this program is to show that every simply connected p–compact group splits into a product of simply connected simple p–compact groups. The proof of this splitting theorem is based on the fact that every classifying space of a p–compact group splits into a product if the normalizer of the maximal torus does.

1. Introduction.

A *loop space* is a triple $X = (X, BX, e : \Omega BX \xrightarrow{\simeq} X)$, where X and BX are topological spaces, BX is pointed, and e is a homotopy equivalence between the loop space ΩBX of BX and X. Such a triple is called a *p–compact group*, if X is $\mathbb{F}_p$–finite, i.e. $H^*(X;\mathbb{F}_p)$ is finite and BX is a pointed p–complete connected space. This notion was introduced by Dwyer and Wilkerson in [7]. Examples of p–compact groups are given by the completion of a compact connected Lie group. For a compact connected Lie group G, the triple $(G^\wedge_p, BG^\wedge_p, \Omega BG^\wedge_p \simeq G^\wedge_p)$ is a p–compact group.

In recent work of Dwyer and Wilkerson [7] [8] and of Møller and the author [15] it turned out that p–compact groups are a good homotopy theoretic generalization of compact Lie groups. In particular, it was shown that p–compact groups enjoy quite a lot of the properties of compact Lie groups; e.g. there always exist maximal tori, normalizers of the maximal tori and Weyl groups [7]. The maximal torus of a p–compact group X is a map $BT_X \to BX$ of an Eilenberg–Mac Lane space $BT_X \simeq K((\mathbb{Z}^\wedge_p)^n, 2)$ into the classifying space BX of X with certain properties. The Weyl group, denoted by W_X, is a finite group, and the normalizer of the maximal torus is a map $BN(T_X) \to BX$, where $BN(T_X)$ fits into a fibration $BT_X \to BN(T_X) \to BW_X$ such that the triangle

$$\begin{array}{ccc} BT_X & \longrightarrow & BN(T_X) \\ & \searrow & \swarrow \\ & BX & \end{array}$$

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commutes up to homotopy. For exact definitions of these notions see Section 2. The space $BN(T_X)$ establishes a loop space $N(T_X) := (N(T_X), BN(T_X), \Omega BN(T_X) \simeq N(T_X))$, where $N(T_X)$ is $\mathbb{F}_p$–finite. In general, this loop space is not a p–compact group, because the space BN_X is not p–complete (the fundamental group $\pi_1(BN_X)$ might not be a finite p–group).

There exists a subloop space $P_X \subset N(T_X)$, which we construct by restricting the fibration to the classifying space of the p–Sylow subgroup of W_X. The classifying space BP_X is p–complete, because $\pi_1(BP_X)$ is a finite p–group and because BT_X is p–complete. This follows from [4; II 5.1, 5.2]. Hence, the loop space P_X is actually a p–compact group. The map $BP_X \longrightarrow BX$ is called the Sylow p–toral subgroup of X and plays the same role as the Sylow p–toral subgroup of a compact Lie group (see 2.9).

The naive approach to believe that the analogy between p–compact groups and compact Lie groups is as good as possible produces a lot of 'theorems' and conjectures. This is done by translating all the notions into the language of p–compact groups, i.e to express everything in terms of classifying spaces, e.g. a *homomorphism* $X \longrightarrow Y$ between p–compact groups is a pointed map $BX \longrightarrow BY$ between the classifying spaces, and a p–compact group $X \cong X_1 \times X_2$ *splits into a product* of p–compact groups if there exists a homotopy equivalence $BX \simeq BX_1 \times BX_2$ (these definitions already make sense for loop spaces).

The lack of an equivalent for the Lie algebra is the main missing piece for a complete dictionary. This forces to find 'new' proofs (in terms of homotopy theory) for 'old' results, which also work for the larger class of p–compact groups.

One of the main questions about p–compact groups asks for a classification of these objects. The classification of compact connected Lie groups says that, for every compact connected Lie group, there exists a finite covering which is a product of simple simply connected Lie groups and a torus. In [15] was shown that the first part of this result is true for connected p–compact groups, namely every connected p–compact group has a finite covering which is a product of a simply connected p–compact group and a torus. For the second step one has to show that every simply connected p–compact group is a product of simple simply connected p–compact groups. A 'new' proof of this second step is the main purpose of this paper.

We call a p–compact group X *simply connected* if the space X is simply connected. The definition of *simple* has to wait until Section 2. It will be given in terms of the Weyl group data and coincides with the traditional definition.

1.1 Theorem. *Let p be an odd prime. Let X be a simply connected p–compact group. Then, $X \cong X_1 \times ... \times X_n$ splits into a product of simple simply connected p–compact groups X_i. The splitting is unique up to isomorphisms and up to the order.*

There exists a notion of a center of a p–compact group[8] [15], which is the generalization of the group or Lie group theoretic center. The center is always a p–compact group. Again, for details see Section 2. A p–compact group X is called centerfree if the center $Z(X)$ of X is the trivial group, i.e. the classifying space $BZ(X)$ is contractible.

1.2 Theorem. *Let p be an odd prime. Let X be a centerfree connected p–compact group. Then, $X \cong X_1 \times ... \times X_n$ splits into a product of simple centerfree connected p–compact groups X_i. The splitting is unique up to isomorphisms and up to the order.*

The proofs of both theorem are based on a general splitting criteria for p–compact groups.

1.3 Theorem. *Let X be a p–compact group and let N_X be the normalizer of a maximal torus $T_X \to X$ of X. If $N_X \cong N_1 \times N_2$ splits into a product (as loop spaces) then $X \cong X_1 \times X_2$ splits into a product of p–compact groups such that $N_{X_i} \cong N_i$ for $i = 1, 2$.*

Remark. The isomorphism $N_{X_i} \cong N_i$ of the above theorem can be made very explicitly . It is induced by the inclusion $X_i \to X$. This will be explained in detail in Section 5. Based on this, a more detailed version of Theorem 1.3 is given in Theorem 5.5.

For the proof of Theorem 1.3 we have to study maps from classifying spaces into almost $B\mathbb{Z}/p$–local spaces. A space A is called *$B\mathbb{Z}/p$–local* if the adjoint $A \to \mathrm{map}(B\mathbb{Z}/p, A)$ of the projection $A \times B\mathbb{Z}/p \to A$ on the first factor is an equivalence and called *almost $B\mathbb{Z}/p$–local* if the adjoint $A \to \mathrm{map}(B\mathbb{Z}/p, A)_{\mathrm{const}}$ into the component of the constant map is an equivalence. Examples of $B\mathbb{Z}/p$–local and almost $B\mathbb{Z}/p$–local spaces are provided by $\mathbb{F}_p$–finite spaces. Let K be a p–complete $\mathbb{F}_p$–finite space. Then, K is $B\mathbb{Z}/p$–local, and $BHE(K)$ is almost $B\mathbb{Z}/p$–local. This follows from the Sullivan conjecture [13] and from [15; 4.13]. Here, $HE(K)$ denotes the monoid of self equivalences of K. In general, this is not an $\mathbb{F}_p$–finite space. Moreover, if K is p–complete and a loop space, the classifying space $BHE(K)$ is also p–complete [15; 4.13]. These are the main examples to we apply the following theorem.

1.4 Theorem. *Let $P \to X$ be a Sylow p–toral subgroup of a p–compact group X and let A be a p–complete almost $B\mathbb{Z}/p$–local space.*

1. *A map $f : BX \to A$ is null homotopic if and only if the restriction $f|_{BP}$ is null homotopic.*
2. *The map $A \to map(BX, A)_{\mathrm{const}}$ is an equivalence.*
3. *If A is $B\mathbb{Z}/p$–local, the map $A \to map(BX, A)$ is an equivalence.*

Remark. Similar results for maps between classifying spaces of p–compact groups may be found in [14] (for (1)) and in [8] (for (2) and for (3) for p–complete $\mathbb{F}_p$–finite spaces as targets).

Theorem 1.4 is one of the basic ingredients of the proof of Theorem 1.3. In the case of classical compact Lie groups, a splitting of the normalizer $N(T_G) \cong N_1 \times N_2$ of the maximal torus $T_G \subset G$ of a compact Lie group G establishes a splitting $W_G \cong W_1 \times W_2$ of the Weyl group and a splitting $T_G \cong T_1 \times T_2$ of the maximal torus such that W_1 acts trivially on T_2. The centralizer $C_G(T_1) \hookrightarrow G$ is rather the second component of G. In fact, it splits into a product of a p–toral group and a subgroup $i_2 : G_2 \subset G$ realizing N_2 as the normalizer of a maximal torus of G_2. Analogously we can construct a second subgroup $i_1 : G_1 \hookrightarrow G$ realizing N_1. Then we construct a map $G_1 \times G_2 \to G$ by showing that $C_G(G_1) \cong G_2$. To prove this isomorphism (for p–compact groups) we have to pass to classifying spaces and to show that $BZ(G_1)^\wedge_p \times BG_2{}^\wedge_p \simeq \mathrm{map}(BG_1, BG^\wedge_p)_{Bi_1}$. Here, $Z(G_1)$ denotes the center of G_1. This is proved in several steps. First, the fiber $(G/G_1)^\wedge_p$ of the fibration $Bi_1 : BG_1{}^\wedge_p \to BG^\wedge_p$ is proved to be homotopy equivalent to $G_2{}^\wedge_p$. Let $P_1 \subset G_1$ be a Sylow p–toral subgroup. Using the splitting of the normalizer and the equivalence $G/G_1 \simeq G_2$, one can show that the restriction to BP_1 is fiber homotopic trivial. With the help of Theorem 1.4 we can establish a triviality criterion for fibrations over classifying spaces (see Section 4). In our situation it says that under certain hypothesis a fibration over BG_1 is fiber homotopically trivial if and only if the restriction to BP_1 is. In particular, this proves that the restriction to BG_1 of the above fibration over BG is trivial and establishes a homotopy pull back diagram

$$\begin{array}{ccc} B{G_1}^\wedge_p \times {G_2}^\wedge_p & \longrightarrow & B{G_1}^\wedge_p \\ \downarrow & & \downarrow \\ B{G_1}^\wedge_p & \longrightarrow & BG^\wedge_p \end{array} .$$

The top horizontal and the top vertical arrow are given by the projection on the first factor. Applying the functor $\mathrm{map}(BG_1, -)$ yields again a pull back diagram. All involved mapping spaces but $\mathrm{map}(BG_1, BG)_{Bi_1}$ can be calculated. These calculation are based on the Sullivan conjecture and on results of [11] (for compact Lie groups) and of [8] (for p–compact groups). The pull back property then proves that $BZ(G_1) \times B{G_2}^\wedge_p \simeq \mathrm{map}(BG_1, BG^\wedge_p)_{Bi}$. Passing to the adjoint establishes the homotopy equivalence $B{G_1}^\wedge_p \times B{G_2}^\wedge_p \simeq BG^\wedge_p$.

Actually, the proof of Theorem 1.3 is much more complicated and uses an induction principle of [8] for p–compact groups, which is based on the "size" of a p–compact group (see the sections 4 and 5).

To prove Theorem 1.2 we have to analyze the normalizers of maximal tori of centerfree p–compact groups and to show that they split into factors associated to simple p–compact groups. This is based on the vanishing of some low dimensional cohomology groups of the Weyl group. Using the analogue of the fibration sequence

$$BZ(G) \to BG \to B(G/Z(G) \to B^2Z(G),$$

for a connected compact Lie group G, Theorem 1.1 is a consequence of Theorem 1.2. Here, $Z(G)$ denotes the center of G, and $G/Z(G)$ is a centerfree compact connected Lie group. This is done in section 6.

Section 2 contains material about p–compact groups, mostly from [7] with some auxiliary lemmas necessary for the proof of Theorem 1.3 and Section 3 contains a calculation of some low dimensional cohomology groups of pseudo reflection groups. The proof of Theorem 1.4 is carried out in Section 4, and the proof of Theorem 1.3 in Section 5. The final section is devoted to the proofs of Theorem 1.1 and Theorem 1.2.

The p–adic rational cohomology $H^*({S^n}^\wedge_p, \mathbb{Q}^\wedge_p)$ is non trivial in infinite high dimensions [4; VI 5.7]. To avoid problems caused by this fact, we define $H^*_{\mathbb{Q}^\wedge_p}(Y) := H^*(Y; \mathbb{Z}^\wedge_p) \otimes \mathbb{Q}$ and use this as cohomology with $\mathbb{Q}^\wedge_p$ as coefficients.

One remark about references. There is some overlap between the papers [7] and [15]. For most citation referring to one of these papers one could also use the other one. We used the one which was first at hand.

Finally, we point out that, quite independently, Dwyer and Wilkerson obtained similar results. In particular, they proved Theorem 1.1 and Theorem 1.2 for all primes.

2. Background.

In this section we recall the basic notions about p–compact groups from [7]. Most of the notions are motivated by classical Lie group theory and by passing to classifying spaces. Because the analogy to compact Lie groups is discussed in [7], [8] and [15], we omit motivations.

2.1 Isomorphisms, monomorphisms, subgroups, conjugation and exact sequences: A homomorphism $f : Y \to X$ of p–compact groups or loop spaces is an *isomorphism* if $Bf : BY \to BX$ is an equivalence. It is a *monomorphism* if the homotopy fiber of the map Bf, denoted by X/Y is $\mathbb{F}_p$–finite. A subgroup $Y \to X$ of p–compact group X is a monomorphism of p–compact groups.

Two homomorphisms $f, g : Y \to X$ are conjugate if the maps Bf and Bg are freely homotopic.

A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of p–compact groups or loop spaces is *short exact* if the associated sequence $BX \xrightarrow{Bf} BY \xrightarrow{Bg} BZ$ is a fibration up to homotopy.

2.2 Special p–compact groups and loop spaces : A *p–compact torus* is a p–compact group $(T, BT, \Omega BT \simeq T)$, where $BT \simeq K((\mathbb{Z}_p^\wedge)^n, 2)$ is an Eilenberg–Mac Lane space in dimension 2.

A *finite group* or a *finite loop space* is a triple $(K, BK, \Omega BK \simeq K)$ such that BK is an Eilenberg–Mac Lane space of a finite group K in dimension 1.

A *finite extension* of a p–adic torus T is a triple $(N, BN, \Omega BN \simeq N)$ which fits into a short exact sequence of loop spaces $T \to N \to W =: N/T$, where W is a finite loop space. A *p–compact toral group* P is a finite extension of a p-compact torus T such that the quotient P/T is a finite p–group. In particular, every p–compact toral group is a p–compact group.

The *component* X_0 *of the unit* of a p–compact group X is given by the component of the constant loop of $\Omega BX \simeq BX$. The classifying space BX_0 is the universal cover of BX and there exists an short exact sequence $X_0 \to X \to \pi_0(X)$ of p–compact groups.

2.4 Centralizers and centers : For a homomorphism $f : Y \to X$ between p–compact groups (or loop spaces), we define the *centralizer* $C_X(f)$ (or $C_X(Y)$ to be the loop space given by the triple

$$C_X(f) := (\Omega \mathrm{map}(BY, BX)_{Bf}, \mathrm{map}(BY, BX)_{Bf}, id) \ .$$

The evaluation at the basepoint $ev : \mathrm{map}(BY, BX)_{Bf} \to BX$ establishes a homomorphism $C_X(f(Y)) \to X$ of loop spaces.

If Y is a p–compact toral group the centralizer $C_X(f)$ is again a p–compact group and the evaluation $C_X(f) \to X$ is a monomorphism [7; 5.1, 5.2 and 6.1]. Moreover, every monomorphism $h : X \to X'$ of p–compact groups induces a monomorphism $C_X(f) \to C_{X'}(hf)$.

A subgroup $Z \to X$ of a p–compact group X is called *central* if the monomorphism $C_X(Z) \to X$ is an isomorphism. The *center* $Z(X)$ of X is the maximal central subgroup of X [8; 1.2] [15; 4.3, 4.4]. To give an explicit definition we use a result of Dwyer and Wilkerson [8; 1.3]. For every p–compact group X, the centralizer $C_X(X)$ is a p–compact group, in fact a product of a p-compact torus and a finite abelian p–group, and $Z(X) := C_X(X) \to X$ is the center of X. For every

p–compact group X there exists a short exact sequence $Z(X) \to X \to X/Z(X) =: PX$ of p–compact groups, and, if X is connected, the quotient PX has a trivial center [15; 4.7].

We call a p–compact group X *centerfree* if $Z(X)$ is the trivial group.

2.5 Maximal tori : The *maximal torus* of a p–compact group X is a monomorphism $T_X \to X$ of a p-compact torus into X such that the centralizer $C_X(T_X)$ is a p-compact toral group, whose component of the unit is given by T_X.

2.6 Theorem [7; 8.11, 8.13 and 9.1]. *Let X be a p–compact group.*

(1) *The p–compact group X has a maximal torus $T_X \to X$.*
(2) *Any subtorus $T \to X$ of X is subconjugate to the maximal torus $T_X \to X$.*
(3) *Any two maximal tori of X are conjugate.*
(4) *If X is connected then $T_X \to C_X(T_X)$ is an isomorphism.*
(5) *X is connected if and only if every finite cyclic subgroup $\mathbb{Z}/p^n \to X$ of X is subconjugate to T_X.*

2.7 Weyl spaces and Weyl groups: Let $T_X \to X$ be a maximal torus of a p–compact group. We convert $BT_X \to BX$ into a fibration. The Weyl space $\mathcal{W}_T(X)$ is defined to be the mapping space of all fiber maps over the identity on BX. Then each component of $\mathcal{W}_T(X)$ is contractible and the Weyl group $W_T(X) := \pi_0(\mathcal{W}_T(X))$ is a finite group under composition [7; 9.5].

2.8 Theorem [7; 9.5 and 9.7]. *Let $T_X \to X$ be the maximal torus of a connected p–compact group X.*

(1) *The action of W_X on BT_X induces representations*

$$W_X \to Aut(H^2_{\mathbb{Q}_p^\wedge}(BT_X)) \cong Gl(n, \mathbb{Q}_p^\wedge)$$

and

$$W_X \to Aut(H_2(BT_X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}) \cong Gl(n, \mathbb{Q}_p^\wedge)$$

which are monomorphisms whose images are generated by pseudo reflections.

(2) *The map $H^*_{\mathbb{Q}_p^\wedge}(BX) \to H^*_{\mathbb{Q}_p^\wedge}(BT_X)^{W_X}$ is an isomorphism.*

2.9 Normalizers, p–normalizers of maximal tori and Sylow p–toral subgroups : Let $i : T_X \to X$ be a maximal torus of a p–compact group X. Again we convert $BT_X \to BX$ into a fibration. The Weyl space $\mathcal{W}_X$ acts on BT_X via fiber maps. This establishes a monoid homomorphism $\mathcal{W}_X \to HE(BT_X)$ where $HE(BT_X)$ denotes the monoid of all self equivalences of BT_X. Passing to classifying spaces establishes a map $B\mathcal{W}_X \to BHE(BT_X)$ which can be thought of as being a classifying map of the fibration $BT_X \to BN(T_X) \to B\mathcal{W}_X$. The total space gives the classifying space of the normalizer $N(T_X)$ of T_X. This is always a finite extension of the p-compact torus T_X (see 2.2).

Let $\mathcal{W}_p$ be the union of those components of $\mathcal{W}_X$ corresponding to a p–Sylow subgroup W_p of W_X. The restriction of the above construction to $\mathcal{W}_p$ gives the classifying space of a p–normalizer $N_p(T_X)$ which we also denote by P_X.

Since the action of $\mathcal{W}_X$ respects the map $BT_X \to BX$, the monomorphism $T_X \to X$ extends to a loop map $N(T_X) \to X$. A p–normalizer fits into an exact sequence $T_X \to N_p(T_X) \to W_p$ and is therefore a p-compact toral group. The restriction $N_p(T_X) \to X$ is a monomorphism [7; 9.9] and is a Sylow p–toral subgroup as the next statement shows.

2.10 Proposition. *Let $P_X \to X$ be a Sylow p–toral subgroup of a p–compact group X as constructed above. Then, the following holds:*

1. *Every p–compact toral subgroup $P' \to X$ of X is subconjugate to P_X.*
2. *The induced map $H^*(BX, \mathbb{F}_p) \to H^*(BP_X; \mathbb{F}_p)$ is a monomorphism.*

Proof. The Euler characteristic of X/P_X is coprime to p [7; proof of Theorem 2.3]. Therefore, part (1) follows from [7; 2.14] and part (2) from [7; 9.8]. □

Let $P_X \to X$ be a Sylow p–toral subgroup of a p–compact group X. Then, by Proposition 2.10, the Sylow p–toral subgroup $P_{X_0} \to X_0$ is subconjugate to P.

2.11 Lemma. *The homomorphism $\pi_0(P_X) \to \pi_0(X)$ is an epimorphism with kernel $\pi_0(P_{X_0})$. There exists a homotopy commutative diagram*

$$\begin{array}{ccccc} BP_{X_0} & \longrightarrow & BP_X & \longrightarrow & B\pi_0(X) \\ \downarrow & & \downarrow & & \| \\ BX_0 & \longrightarrow & BX & \longrightarrow & B\pi_(X), \end{array}$$

where the rows are fibration sequences.

Proof. This follows from [15; 3.8 and 3.9]. □

2.12 Simply connected and simple p–compact groups : A p–compact group X is called *simply connected*, if X is simply connected, and X is called *simple*, if the associated representation $W_X \to Aut(H_2(BT_X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q})$ is irreducible. By [5], this is equivalent to the fact that the associated complex representation is irreducible. The notion of simple is motivated by the classical situation. For a compact connected Lie group G, the representation $W_G \to H_2(BT_G; \mathbb{C})$ is irreducible if and only if G is simple.

2.13 Elementary abelian subgroups: Let X be a p–compact group, and let $i : P \to X$ be a Sylow p–toral subgroup. Let $j : E \to X$ be a monomorphism of an elementary abelian group E into X. By 2.10, the subgroup E is subconjugate to P via a homomorphism $j' : E \to P$. Such a subconjugation is called *special* if $C_P(j') \to C_X(j)$ is a Sylow p–toral subgroup.

2.14 Lemma.

1. *For every monomorphism $j : E \to X$, there exists a special subconjugation $j' : E \to P$.*
2. *Let*

$$\begin{array}{ccc} E & \xrightarrow{\ \alpha\ } & E_1 \\ & {\scriptstyle j}\searrow \quad \swarrow{\scriptstyle j_1} & \\ & X & \end{array}$$

be a diagram commuting up to conjugation. and let $j' : E \to P$ be a special subconjugation of j. Then, there exists a special subconjugation $j_1' : E_1 \to P$ such that

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & E_1 \\ & {\scriptstyle j'}\searrow \quad \swarrow{\scriptstyle j_1'} & \\ & P & \end{array}$$

commutes up to conjugation.

Proof. Let $l : P' \to C_X(j)$ be a Sylow p–toral subgroup. Because E is a central subgroup of $C_X(j)$ [7; 8.2], it is also contained in P' as a central subgroup [8; 3.2, 5.5]. By Proposition 2.10, there exists a subconjugation $k : P' \to P$. The composition $j' : E \xrightarrow{j} P' \xrightarrow{k} P$ is a subconjugation of j into P. We get a sequence of monomorphisms

$$P' \xrightarrow{\cong} C_{P'}(j') \to C_P(j) \to C_X(j).$$

The first arrow is an isomorphism, because E is central in P', the second arrow is induced by k, and the third arrow by i. The composition is conjugate to l. The centralizer $C_P(j)$ is a p-compact toral group and therefore subconjugate to $l : P' \to C_X(j)$ via a subconjugation $m : C_P(j) \to P'$. Because a self map of a p–compact group is an isomorphism if it is a monomorphism [15; 3.4], we have $P' \cong C_P(j'(E))$. Hence, j' is special. This proves part (1).

Let $E_2 \subset E_1$ be a complement of E, i.e. $E_1 \cong E \oplus E_2$. Taking the adjoint of $j_1 : E \oplus E_2 \to X$ establishes map $j_2 : E_2 \to C_X(j(E))$. Because j' was special, $P' := C_P(j'(E)) \to C_X(j(E))$ is a Sylow p–toral subgroup. Applying part (1) yields a special subconjugation $j_2' : E_2 \to P'$, and passing back to maps into P and X proves part (2). □

The elementary abelian subgroups of a p–compact group X make up a category $A_p(X)$, which we call the Quillen category of X. An object of $A_p(X)$ is a monomorphism $E \xrightarrow{i_E} X$, where E is a nontrivial elementary abelian group, and a morphism is a triangle

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

which commutes up to conjugation. Every morphism $(E_1 \to X) \to (E_2 \to X)$ establishes a map $BC_X(E_2) \to BC_X(E_1)$. Hence, setting $\phi(E \to X) := BC_X(E)$ defines a functor

$$\phi : A_p(X) \to \mathcal{T}op.$$

Evaluation at base points defines a map

$$\Phi : \underset{A_p(X)}{\underrightarrow{\text{hocolim}}}(\phi) \to BX \ .$$

Let $\phi^* : A_p(X) \to \mathcal{A}b$ be the functor given by $\phi^*(E \to X) := H^*(BC_X(E); \mathbb{F}_p)$. The following theorem is a collection of results of [8; 8.1 and the proof, 8.2].

2.15 Theorem [8]. *Let X be a p–compact group.*

(1) *The category $A_p(X)$ is mod–p acyclic, i.e. for the constant functor $F_{\mathbb{Z}/p}$ taking as value $\mathbb{Z}/p$, we have*

$$\varprojlim_{A_p(X)}{}^{i} F_{\mathbb{Z}/p} = \begin{cases} 0 & \textit{for } i > 0 \\ \mathbb{Z}/p & \textit{for } i = 0 \end{cases}$$

(2) *The map $\Phi : \operatorname{hocolim}_{\overrightarrow{A_p(X)}} (\phi) \to BX$ is a homotopy equivalence.*

(3) *We have*

$$\varprojlim_{A_p(X)}{}^{i} \phi^* = \begin{cases} 0 & \textit{for } \ i > 0 \\ H^*(BX; \mathbb{F}_p) & \textit{for } \ i = 0 \end{cases}$$

2.16 Cohomological dimension: The *cohomological dimension* $\mathrm{cd}_{\mathbb{F}_p}(Y)$ of a $\mathbb{F}_p$–finite space Y is defined to be the dimension of the top non vanishing mod-p cohomology group.

2.17 Lemma. *Let $T \to X$ be a subtorus of a p–compact group X. Let X_0 and C_0 denote the components of the unit of X and $C := C_X(T)$. Then the following holds:*

(1) *$C \to X$ is a subgroup of maximal rank.*
(2) *$C_0 \cong C_{X_0}(T)$ and $\pi_0(C) \to \pi_0(X)$ is an injection.*
(3) *If $W_C \cong W_X$ then $C \to X$ is an isomorphism of p–compact groups.*
(4) *If T is a non central subgroup of X_0, then $cd_{\mathbb{F}_p}(C) < cd_{\mathbb{F}_p}(X)$.*

Proof. Part (1) is obvious, because every subtorus is subconjugate to X_0 and to every maximal torus (Theorem 2.6). Because $C_{X_0}(T)$ is connected [15; 3.11], we have a sequence $C_{X_0}(T) \subset C_0 \subset X_0$ of subgroups of maximal ranks. Since C_0 centralizes T, we have $C_0 \subset C_{X_0}(T)$. Hence, this inclusion induces an isomorphism between the Weyl groups and is therefore an isomorphism of connected p–compact group [15; 3.7]. This proves the first part of (2). The second part of (2) is now obvious.

Part (2) establishes a commutative diagram

$$\begin{array}{ccccc} C_0 & \longrightarrow & C & \longrightarrow & \pi_0(C) \\ \downarrow & & \downarrow & & \downarrow \\ X_0 & \longrightarrow & X & \longrightarrow & \pi_0(X) \ . \end{array}$$

By [15; 3.8], we have $\pi_0(X) \cong W_X/W_{X_0}$ for every p–compact group. Hence, if $W_C \cong W_X$, then the right vertical arrow is an epimorphism and therefore an isomorphism. This implies that $W_{C_0} \cong W_{X_0}$, that $C_0 \cong X_0$ [15; 3.11] and that $C \cong X$, which is part (3).

If $\mathrm{cd}_{\mathbb{F}_p}(C_X(T)) = \mathrm{cd}_{\mathbb{F}_p}(X)$, then $\mathrm{cd}_{\mathbb{F}_p}(X_0) = \mathrm{cd}_{\mathbb{F}_p}(X) = \mathrm{cd}_{\mathbb{F}_p}(C) = \mathrm{cd}_{\mathbb{F}_p}(C_{X_0}(T))$. Because every monomorphism between connected p–compact groups of the same cohomological dimension is an isomorphism [7; 6.14, 6.15], this implies that the left vertical arrow is an isomorphism and that $T \to X_0$ is a central subgroup. This is a contradiction and proves the last part. $\square$

We finish this section with two results, necessary for later purpose.

2.18 Lemma.
(1) Let $Bf : BX \times BY \to BX$ be a map between classifying spaces of p–compact groups such that $Bf|_{BX} \simeq id$ and $Bf|_{BY} \simeq *$. Then, Bf is homotopic to the projection $pr_1 : BX \times BY \to BX$ on the first factor.

(2) Let $Bg : \prod_i BX_i \to \prod_i BX_i$ be a self map of a product of classifying spaces p–compact groups such that the compositions $Bg_{k,l} : X_l \to \prod_i X_i \to \prod_i X_i \to X_k$ are null homotopic for $k \neq l$ and such that $Bg_{k,k}$ is an equivalence. Then, $Bg \simeq \prod_k Bg_{k,k}$.

Proof. The adjoint of Bf gives a map $BX \to \mathrm{map}(BY, BX)_{\mathrm{const}} \xrightarrow{\simeq} BX$. The second map is given by evaluation at the basepoint and is an equivalence (Theorem 1.4 or [8; 10.1]). By assumption, the composition is homotopic to the identity and the adjoint is therefore homotopic to the projection pr_1. This proves the first part.

The product $Bg' := \prod_k Bg_{k,k}$ is an equivalence of p–compact groups. Considering the composition $Bg'^{-1}Bg$ shows that we can assume that $Bg_{k,k}$ is homotopic to the identity. Now we first prove part (2) for two factors $BX_1 \times BX_2$. By part (1) the composition $BX_1 \times BX_2 \xrightarrow{\phi} BX_1 \times BX_2 \xrightarrow{\mathrm{pr}_i} BX_i$ is homotopic to the projection on the i–factor. This implies that $B\phi \simeq \mathrm{id}$. In general, the second part is proved by an induction over the number of factors. □

2.19 Proposition. *Let X be a p–compact group. Let $i_N : N \to X$ be the normalizer of a maximal torus $i_T : T \to X$ and $i_P : P \to N \to X$ a Sylow p–toral subgroup. Then, the maps $BZ(P) \to map(BP, BP)_{id} \to map(BP, BX)_{Bi_P}$ and $map(BN, BN)_{id} \to map(BN, BX)_{Bi_N}$ are homotopy equivalences.*

Proof. The first equivalence follows from [8; 1.2]. The inclusion i_T factors over the inclusion $i : T \to P$. If we convert the map $BT \to BP$ into a fibration respectively a covering, then we get a free action of the finite group $\overline{P} := P/T$ on BT. By [10; 5.1], there exists a natural transformation of functors $\mathrm{map}(BP, \ ) \to \mathrm{map}(BT, \ )^{h\overline{P}}$, which for every target is an equivalence. Here, $\overline{P}$ acts on $\mathrm{map}(BT, \ )$ via the action on BT and, for every $\overline{P}$–space Y, the homotopy fixed–point set $Y^{h\overline{P}}$ is given by the space of sections of the Borel construction $E\overline{P} \times_{\overline{P}} Y \to B\overline{P}$. The map $\mathrm{map}(BT, BP)_{Bi} \to \mathrm{map}(BT, BX)_{Bi_T}$ is an equivalence because T is a maximal torus and because $C_X(T)$ is subconjugate to P (Proposition 2.10). Moreover, this map is $\overline{P}$–equivariant, and induces therefore an equivalence $\mathrm{map}(BT, BP)^{h\overline{P}}_{Bi} \to \mathrm{map}(BT, BX)^{h\overline{P}}_{Bi_T}$. The component of $\mathrm{id} : BP \to BP$ is obviously mapped onto the component $Bi_P : BP \to BX$. This establishes the second equivalence.

For the last equivalence we use the same trick. We convert the map $Bj : BT \to BN$ into a covering with the Weyl group W of X as the deck transformation group. We only have to show that $\mathrm{map}(BT, BN)_{Bj} \to \mathrm{map}(BT, BX)_{Bi_T}$ is an equivalence. But this follows from [17; 3.7] and [8; 7.6]. Both mapping spaces are the classifying spaces of a finite extensions of T whose group of components is given by the Weyl group elements acting trivially on T. □

3. Low dimensional cohomology groups of pseudo reflection groups.

The main result of this chapter states vanishing results for some cohomology groups of pseudo reflection groups at odd primes. Let U be a finite dimensional vector space over the p–adic rationales. An element of $Gl(U)$ is a *pseudo reflection* if it is of finite order and fixes a hyperplane. A faithful representation $\rho : W \to Gl(U)$

of a finite group W is called a *pseudo reflection group*, if the image $\rho(W)$ is generated by pseudo reflections. The representation is called a *honest real reflection group* or a Weyl group, if $\rho(W)$ is generated by honest reflections and if the representation is already defined over $\mathbb{Q}$. A pseudo reflection group is called *irreducible*, if the representation is irreducible. We say that a finite group W is a pseudo reflection group, if there exists a representation taking W to a pseudo reflection group.

3.1 Proposition. *Let W be a pseudo reflection group. Then, for an odd prime p, the homology and cohomology groups with trivial coefficients $H_1(W;\mathbb{Z}_p^\wedge)$, $H_1(W;\mathbb{F}_p)$, $H_2(W;\mathbb{Z}_p^\wedge)$, $H_2(W;\mathbb{F}_p)$, $H^1(W;\mathbb{Z}_p^\wedge)$, $H^1(W;\mathbb{F}_p)$, $H^2(W;\mathbb{Z}_p^\wedge)$, $H^2(W;\mathbb{F}_p)$, $H^3(W;\mathbb{Z}_p^\wedge)$ all vanish.*

Proof. Because W is a finite group, it is sufficient to look at $H_1(W;\mathbb{Z}_p^\wedge)$ and $H^2(W;\mathbb{Z}/p)$. Then, for the other groups, the statement follows by universal coefficient theorems. Because every pseudo reflection group splits into a product of irreducible pseudo reflection groups and because of the Künneth–formula we also can assume that W is irreducible.

The group $H_1(W;\mathbb{Z})$ is isomorphic to the abelianization of W. Because W is generated by elements of order dividing $p-1$, this is a finite abelian group of order coprime to p. Thus, $H_1(W;\mathbb{Z}_p^\wedge)=0$.

The cohomology group $H^2(W;\mathbb{F}_p)$ classifies central extensions of the form $\mathbb{Z}/p\to N\to W$. We want and we have to show that every central extension of this form splits.

First we consider the case of an honest real reflection group. Let $R\subset W$ denote the set of reflections of W and $B\subset R$ a minimal set of generators of W. Let $\sigma_1,...\sigma_n$ denote the elements of B. Then there exist integers $m_{i,j}$ such that

$$W\cong\langle\sigma_1,...,\sigma_n\rangle/\langle(\sigma_i\sigma_j)^{m_{i,j}}:1\le i,j\le n\rangle$$

is the quotient of the free group generated by the elements of B dividing out only relations of the form $(\sigma_i\sigma_j)^{m_{i,j}}$ [3; Chap. 5]. Let $\hat{s}:W\to N$ be a set theoretic section. We define $s:R\to N$ by $s(\sigma):=\hat{s}(\sigma)^p$. Because the extension is central, the map s does not depend on $\hat{s}$. If $\hat{s}$ was a homomorphism, then $\hat{s}(\sigma)=s(\sigma)$. Hence there is at most one group theoretic section. This also follows from the vanishing of $H^1(W;\mathbb{F}_p)$.

We show that s can be extended to a group theoretic section $W\to N$; i.e. s has to satisfy the relations $(s(\sigma_i)s(\sigma_j))^{m_{i,j}}=1$ for all $1\le i,j\le n$.

Let $W'\subset W$ be the subgroup generated by σ_1 and σ_2, and let $N'\subset N$ be the subgroup defined by the pull back diagram

$$\begin{array}{ccccc}\mathbb{Z}/p & \longrightarrow & N' & \longrightarrow & W'\\ \| & & \downarrow & & \downarrow\\ \mathbb{Z}/p & \longrightarrow & N & \longrightarrow & W\end{array}\quad.$$

By the classification list of pseudo reflection groups [5] there exist only three reflection groups generated by two honest reflections, namely $\mathbb{Z}/2\times\mathbb{Z}/2$, Σ_3 and the dihedral group D_{12} of 12 elements. A short calculation shows that in all cases $H^2(W',\mathbb{Z}/p)=0$. For $\mathbb{Z}/2\times\mathbb{Z}/2$ this is obvious, for Σ_3 one uses the fact that

$\Sigma_3/\mathbb{Z}/3 \cong \mathbb{Z}/2$, and for D_{12} one observes that $\Sigma_3 \subset D_{12}$ is a subgroup of index 2. Therefore there exists a group theoretic section $s' : W' \to N'$. By the independence of s, for $i = 1, 2$, we have $s(\sigma_i) = s'(\sigma_i)^p = s'(\sigma_i^p) = s'(\sigma_i)$. This shows that s satisfies the relation for σ_1 and σ_2, and analogously, all relations. Hence, the map s can be extended to a group theoretic section, the sequence $\mathbb{Z}/p \to N \to W$ splits, and $H^2(W; \mathbb{Z}/p) = 0$ for honest real reflection groups.

If W is a pseudo reflection group, then by the classification list [5], the order of W is coprime to p or W is one of the groups of number 1, 2b for $p = 3$, 28, 35, 36 or 37, which all describe honest real reflection groups, or belongs to one of the numbers 2a, 12, 29, 31 or 34. We refer here to the numbering of [5]. In all the latter cases we only have to consider one prime. The following table indicates this prime and denotes a subgroup of index coprime to p. Moreover, the subgroup is a honest real reflection group.

no.	group order	*prime*	*subgroup*	*index*
$2a$	$r \cdot m^{n-1} \cdot n!$	$p \leq n$	Σ_n	$r \cdot m^{n-1}$
12	48	$p = 3$	Σ_3	8
29	$2^8 \cdot 3 \cdot 5$	$p = 5$	Σ_5	2^5
31	$64 \cdot 6!$	$p = 5$	Σ_5	$3 \cdot 2^7$
34	$108 \cdot 9!$	$p = 7$	Σ_7	$2^5 \cdot 3^5$.

In no. 2a, the number m divides $p - 1$ and r divides m. For no. 2a the information about the subgroup might be found in [19] and for all other numbers in [1]. Therefore, in all these cases, $H^2(W; \mathbb{Z}/p)$ also vanishes. □

The following proposition is needed in Section 6.

3.2 Proposition. *Let p be an odd prime. For $i = 1, 2$ let $W_i \to Gl(U_i)$ be a pseudo reflection group and $L_i \subset U_i$ be a $[W_i]$–lattice.*

(1) *For the trivial action of W_2 on L_1, the map*

$$H^3(W_1; L_1) \to H^3(W_1 \times W_2; L_1)$$

is an isomorphism.

(2) *The map*

$$H^3(W_1; L_1) \oplus H^3(W_2; L_2) \to H^3(W_1 \times W_2; L_1 \times L_2)$$

is an isomorphism.

Proof. Part (1) is a consequence of Proposition 3.1. Again by Proposition 3.1 we have

$$H^2(W_1; H^1(W_2; L_1)) = H^1(W_1; H^2(W_2; L_1)) = H^0(W_1; H^3(W_2; L_1)) = 0 \ .$$

Hence, using a Hochschild–Serre spectral sequence argument, we get $H^3(W_1 \times W_2; L_1) \cong H^3(W_1; H^0(W_2; L_1)) \cong H^3(W_1; L_1)$. This implies part (2). □

4. Proof of Theorem 1.4.

In [8] is set up a induction principle, which we will use in this and the next section, for proving statements about p–compact group.

4.1 Definition. A class Cl of p–compact groups is called *saturated* if it satisfies the following 5 conditions:

1. If $X \in Cl$ and $Y \cong X$, then $Y \in Cl$, i.e. Cl is closed under equivalences.
2. Every p-compact toral group belongs to Cl.
3. If the identity component X_0 of X is in Cl and if any p–compact group Y, such that $\mathrm{cd}_{\mathbb{F}_p}(Y) < \mathrm{cd}_{\mathbb{F}_p}(X)$, is in Cl then X also belongs to Cl.
4. If X is connected, and if $X/Z(X)$ is in Cl, then X is in Cl.
5. If X is connected and centerfree, and $Y \in Cl$ for all p–compact groups such that $\mathrm{cd}_{\mathbb{F}_p}(Y) < \mathrm{cd}_{\mathbb{F}_p}(X)$, then $X \in Cl$.

4.2 Theorem [8; 9.2]**.** *Any saturated class of p–compact groups contains all p–compact groups.*

4.3 Remark. Our definition of a saturated class is not exactly the same as the one Dwyer and Wilkerson give (there are some differences in (2) and (3)), but their argument also works in our situation to prove Theorem 4.2.

Similar results as those of Theorem 1.4 are already proven by Dwyer and Wilkerson and Møller. We apply their arguments in our situation to get a proof of Theorem 1.4.

Proof of Theorem 1.4. Let X and Y be two p–compact groups and let Z be a p–complete $\mathbb{F}_p$–finite space. In [8; 9.3, 10.1] is stated that the map $Z \to \mathrm{map}(BX, Z)$ is an equivalence (which is the Sullivan conjecture for p–compact groups) and that $BY \to \mathrm{map}(BX, BY)_{\mathrm{const}}$ is an equivalence. In [14; §5] is proven that a map $f : BX \to BY$ is null homotopic if and only if the restriction $f|_{BP}$ is null homotopic for a Sylow p–toral subgroup $P \subset X$ of X.

The second assertion is an easy consequence of the first (apply the loop functor to the mapping space of pointed maps and remember that $\Omega BY \simeq Y$ is $\mathbb{F}_p$–finite and p–complete). The first and third statement are proven using the induction principle. That is that the class of p–compact groups satisfying one of the claims is shown to be a saturated class. The main input in both proofs is the classical Sullivan conjecture for $B\mathbb{Z}/p$; i.e. that Y and Z are $B\mathbb{Z}/p$–local.

A space A is almost $B\mathbb{Z}/p$–local if and only if the loop space ΩA is $B\mathbb{Z}/p$–local. Having this in mind, an inspection of the arguments in [8] and [14] shows that both proofs also work in our situation. This shows that the map $A \to \mathrm{map}(BX, A)$ is an equivalence if A is p–complete and $B\mathbb{Z}/p$–local, that $A \to \mathrm{map}(BX, A)_{\mathrm{const}}$ is an equivalence and that a map $f : BX \to A$ is null homotopic if and only if the restriction $f|_{BP}$ is null homotopic for a Sylow p–toral subgroup P of X. For the last two assertions we only have to assume that A is p–complete and almost $B\mathbb{Z}/p$–local. This proves Theorem 1.4. □

We note the following corollary, which we will apply in the next section.

4.4 Corollary. *Let X be a p–compact group with Sylow p–toral subgroup $P \to X$. Let $F \to E \to BX$ be a fibration, such that F is $\mathbb{F}_p$–finite, p–complete and a loop* *space. If the restriction to BP induces a fibration which is fiber homotopy equivalent to the trivial fibration, then the fibration itself is fiber homotopically trivial.*

Proof. By [20], the fibration is classified by a map $BX \to BHE(F)$. The restriction to BP is null homotopic. The space $BHE(F)$ is connected, p–complete and almost $B\mathbb{Z}/p$–local [15; 4.13]. Thus, the statement follows from Theorem 1.4. □

A statement similar to the last corollary is also true if the fiber is the classifying space of a p-compact torus.

4.5 Proposition. *Let T be a p–compact torus, let X be a p–compact group with Sylow p–toral subgroup $P \to X$, and let $BT \to E \to BX$ be a fibration. If the restriction to BP induces a fiber homotopic trivial fibration, then the fibration itself is fiber homotopically trivial.*

The proof is analogous to the proof of Corollary 4.4, but based on the following lemma.

4.6 Lemma. *Let T be a p–compact torus, and let X be a p–compact group with Sylow p–toral subgroup $i : P \to X$. If, for a map $f : BX \to BHE(BT)$, the composition $f \circ Bi$ is null homotopic, then f is null homotopic.*

Proof. Let $P' \to X_0$ be the Sylow p–toral subgroup of the component of the unit. Passing to the first Postnikov section is a coaugmented functor and establishes therefore the diagram

$$\begin{array}{ccccc}
 & & & & BSHE(BT) \simeq B^2T \\
 & & & & \downarrow \\
BP & \xrightarrow{Bi} & BX & \xrightarrow{f} & BHE(BT) \\
\downarrow & & \downarrow & & \downarrow \\
B(P/P') & \xrightarrow{\simeq} & B\pi_0(BX) & \xrightarrow{\bar{f}} & B\pi_0(HE(BT)) \, .
\end{array}$$

Here, $SHE(BT^\wedge_p)$ denotes the monoid of self homotopy equivalences homotopic to the identity and the right column is a fibration sequence. The equivalence $B(P/P') \simeq B\pi_0(BX)$ follows from Lemma 2.11. Because of the equivalence in the bottom line and because $f \circ Bi$ is null homotopic, the map $\bar{f}$ is also null homotopic. As an Eilenberg–MacLane space in dimension 2, the space BT carries a multiplication. The associated adjoint is a map $BT \to SHE(BT)$ which is an equivalence and a loop map and establishes therefore an equivalence $B^2T \xrightarrow{\simeq} BSHE(BT)$. Moreover, up to homotopy, this map is equivariant with respect to the action of π. Thus, homotopy classes of lifts of the compositions $BP \to B\pi_0(X) \to B\pi$ and $BX \to B\pi_0(X) \to B\pi$ to $BHE(BT)$ are classified by the obstruction groups $H^3(BP, \pi_3(BT))$ or $H^3(BX, \pi_3(BT))$ (all other obstruction groups vanish). The existence of a transfer as a stable map [6] shows that the inclusion $BP \to BX$ induces a monomorphism $H^*(BX, M) \to H^*(BP, M)$ for any systems of coefficients. The composition $f \circ Bi$ is null homotopic by assumption and so is the map f. This proves the statement. □

5. Proof of an improved Theorem 1.3.

Let X be a p–compact group with normalizer N_X of a maximal torus $T_X \to X$ of X. Let $N_X \cong N_1 \times N_2$ be a splitting into two factors. For $i = 1, 2$, we first construct a subgroup $Y_i \to X$ related to N_i in the following sense:

5.1 Definition. Let $N \subset N_X$ be a finite extension of a torus. A p–compact subgroup $Y \subset X$ is associated to N if the induced homomorphism $\pi_0(Y) \to \pi_0(X)$ is an injection and if there exists an isomorphism $N \cong N_Y$ such that the diagram of loop spaces

$$\begin{array}{ccc} & N_Y \cong N & \\ \swarrow & & \searrow \\ Y & \longrightarrow & X \end{array}$$

commutes up to conjugation.

For any p–compact group X there exists a fibration

$$BX_0 \to BX \to B\pi$$

where X_0 denotes the connected component of the unit and π the group of the components. The composition $BN_X \to BX \to B\pi$ factors over $BW_X \to B\pi$, which is induced by a homomorphism $W_X \to \pi$. The kernel is given by the Weyl group W_{X_0} [15; 3.8], which is a pseudo reflection group.

5.2 Lemma. *The splitting $N_X \cong N_1 \times N_2$ induces the splittings: $W_X \cong W_1 \times W_2$, $T_X \cong T_1 \times T_2$, $W_{X_0} \cong W_{1,0} \times W_{2,0}$, $N_{X_0} \cong N_{1,0} \times N_{2,0}$ and $\pi := \pi_0(X) \cong \pi_1 \times \pi_2$ of finite groups or loop spaces. All these splittings are related in the sense indicated by the subscripts.*

Proof. For W_X and T_X this is obvious. Every pseudo reflection $\sigma \in W_{X_0}$ is contained either in W_1 or in W_2. Let $W_{i,0} \subset W_{X_0}$ be the subgroup generated by all pseudo reflection contained in W_i, and let $N_{i,0}$ be the counter image of $W_{i,0}$ in N_i. Because W_{X_0} is generated by pseudo reflections (Theorem 2.8), we have splittings $W_{X_0} \cong W_{1,0} \times W_{2,0}$, $N_{X_0} \cong N_{1,0} \times N_{2,0}$ and $\pi \cong \pi_1 \times \pi_2$. $\square$

Now we construct subgroups $Y_i \subset X$ associated to $N_i \subset N_X$. We define $C_1 := C_X(T_2) \subset X$. This is a subgroup of maximal rank. We note the following lemma:

5.3 Lemma.

(1) $N_{C_1} \cong C_{N_X}(T_2) \cong N_1 \times C_{N_2}(T_2)$.
(2) $\pi_0(C_1) \to \pi_0(X)$ *is a monomorphism.*

Proof. The centralizer $C_1 \subset X$ is a subgroup of maximal rank and the Weyl group W_{C_1} is given by all elements of W_X acting trivially on T_2 [8; 7.6]. Hence, $W_{C_1} = W_1 \times W_2'$, where $W_2' = W_{C_1} \cap W_2 \subset W_2$. Let $q : N_X \to W_X$ be the projection. Then, $N_{C_1} = q^{-1}(W_{C_1}) = C_{N_X}(T_2) = N_1 \times C_{N_2}(T_2)$. For the last two expressions, there is a remark in order. The loop spaces N_X and N_2 are not p–compact groups, but, as finite extensions of p–compact tori, behave very much like p–compact groups. In particular, the centralizers $C_{N_X}(T_2)$ as well as $C_{N_2}(T_2)$ are again finite extensions of p–compact tori (see [17; proof of 3.7]). This proves the first part, the second follows from Lemma 2.17. $\square$

By the above lemma, $N_1 \subset N_{C_1}$ and $\pi_1 \subset \pi_0(C_1) \subset \pi_0(X)$. Let $\hat{Y} = \Omega B\hat{Y}$ be the p–compact group defined by the pull back diagram

$$\begin{array}{ccc} B\hat{Y} & \longrightarrow & BC_1 \\ \downarrow & & \downarrow \\ B\pi_1 & \longrightarrow & B\pi_0(C_1)\ . \end{array}$$

By construction, we have $N_{\hat{Y}} = N_1 \times T_2$. Applying the functor map$(BT_2, \ )$ shows that $T_2 \subset \hat{Y}$ is central, because $T_2 \subset C_1$ is. Using [2; 7.2] we can divide out this central subgroup. This establishes the commutative diagram of fibrations

$$\begin{array}{ccccc} BT_2 & \longrightarrow & BT_1 \times BT_2 & \longrightarrow & BT_1 \\ \| & & \downarrow & & \downarrow \\ BT_2 & \longrightarrow & BN_1 \times BT_2 & \longrightarrow & BN_1 \\ \| & & \downarrow & & \downarrow \\ BT_2 & \longrightarrow & B\hat{Y} & \longrightarrow & B(\hat{Y}/T_2) =: BY_1 \end{array} \quad .$$

The composition $T_1 \to Y_1$ is a maximal torus and $N_1 \to Y_1$ is the normalizer of this maximal torus. Actually, all lines are principal fibration (again by [2; 7.2]). Thus, the bottom fibration is classified by a map $BY_1 \to B^2T_2$, which is determined by a cohomology class in $H^3(BY_1; \pi_3(B^2T_2))$. The restriction of this map to BN_1 is homotopically trivial, and the induced map $H^3(BY_1; \pi_3(B^2T_2)) \to H^3((BN_1; \pi_3(B^2T_2))$ is a monomorphism (Proposition 2.10). Hence, the bottom row is fiber homotopy equivalent to the trivial fibration and we can split $B\hat{Y} \simeq BY_1 \times BT_2$ in such a way that we get a commutative diagram

$$\begin{array}{ccc} & BN_1 \times BT_2 & \\ {\scriptstyle Bj \times id} \swarrow & & \searrow \\ B\hat{Y} \simeq BY_1 \times BT_2 & \longrightarrow & B\hat{Y} \end{array} \quad .$$

The p–compact subgroup $Y_1 \subset \hat{Y} \subset X$ is associated to $N_1 \subset N_X$, because $\pi_0(Y_1) \cong \pi_1$. This proves the first part of the following lemma:

5.4 Lemma. *Let X be a p–compact group. Let $N_X \cong N_1 \times N_2$ split into two factors. Then the following holds:*

(1) *There exist p–compact subgroups $Y_1, Y_2 \subset X$ associated to $N_1, N_2 \subset N_X$.*
(2) $\pi_0(C_1) \cong \pi_0(Y_1) \times \pi_0(C_{N_2}(T_2))$
(3) *If X is connected, the subgroups Y_1 and Y_2 are also connected. and there exists an isomorphism $Z(X) \cong Z(Y_1) \times Z(Y_2)$ making the diagram*

$$\begin{array}{ccc} Z(X) & \xrightarrow{\cong} & Z(Y_1) \times Z(Y_2) \\ \downarrow & & \downarrow \\ T_X & \xrightarrow{\cong} & T_1 \times T_2 \end{array}$$

commutative.

Proof. By construction $N_1 \times C_{N_2}(T_2) \cong C_{N_X}(T_2) \longrightarrow C_1$ is the normalizer of the maximal torus $T_X \to C_1$. Hence, the map $N_1 \times C_{N_2}(T_2)$ induces an epimorphism on the group of components, and the kernel is given by $\pi_0(N_{1,0}) \cong W_{1,0}$ [15; 3.8]. The isomorphism $\pi_0(Y_1) \cong W_1/W_{1,0}$ now establishes the equation of part (2).

If X is connected, the centralizer $C_X(T_2)$ is also connected and splits into the product $Y_1 \times T_2$. All this follows from the construction. Hence, Y_1 is connected as well as Y_2. For connected p–compact groups the center is always a subgroup of the maximal torus and can be calculated, only knowing the normalizer of the maximal torus [8; 7.5]. This proves part (2). $\square$

The next theorem is a more detailed version of Theorem 1.3.

5.5 Theorem. *Let X be a p–compact group such that $N_X \cong N_1 \times N_2$. For $i = 1, 2$, let $Y_i \xrightarrow{j_i} X$ be a subgroup associated to N_i. Then there exists an isomorphism $j : Y_1 \times Y_2 \to X$ of p–compact groups, which, for $i = 1, 2$, makes the diagram*

$$\begin{array}{ccc} N_1 \times N_2 & \longrightarrow & N_X \\ \downarrow & & \downarrow \\ Y_1 \times Y_2 & \xrightarrow{\ j\ } & X \\ \uparrow & \nearrow_{j_i} & \\ Y_i & & \end{array}$$

commutative up to conjugation.

Proof. The proof is based on the induction principle of Dwyer and Wilkerson explained in Section 4. For convenience, we collect the assumptions of the theorem in a condition :

A triple $(X; Y_1, Y_2)$ of p–compact groups satisfies *condition (A)*, if $N_X \cong N_1 \times N_2$, and if the subgroups $Y_1, Y_2 \to X$ are associated to N_1 and N_2.

If not specified otherwise, we denote all obvious inclusions of subgroups and subloop spaces by i. We also fix Sylow p–toral subgroups $P_i \to N_i \to Y_i$ of Y_i. The product $P_1 \times P_2 \to N_X \to X$ is then a Sylow p–toral subgroup of X.

Let Cl be the class of all p–compact groups satisfying the statement. We show that Cl is a saturated class.

Step 1: Let X be a p–compact group such that $N_X \cong N_1 \times N_2$. Let $Y \xrightarrow{\cong} X$ be an isomorphism of p–compact groups. Then, the normalizer $N_Y \cong N_X \cong N_1 \times N_2$ also splits. Let $Y_i \to X$ be a subgroup associated to N_i. Then $Y_i \to X \xleftarrow{\cong} Y$ can also be considered as a subgroup of Y associated to N_i. If Y satisfies the theorem, the composition $Y_1 \times Y_2 \cong Y \cong X$ establishes the desired splitting of X. Hence, the class Cl is closed under equivalences.

Step 2 : Theorem 5.5 is obviously satisfied by p–toral groups.

Step 3: Let X be a p–compact group such that $X_0 \in \mathcal{C}l$ and such that each p–compact group with smaller cohomological dimension belongs to $\mathcal{C}l$. We have to show that $X \in \mathcal{C}l$. Let $(X; Y_1, Y_2)$ be a triple of p–compact groups satisfying condition (A). We calculate the mapping space $\mathrm{map}(BY_1, BX)_{Bi_1}$ and show that there exists a map $BY_2 \to \mathrm{map}(BY_1, BX)_{Bi_1}$ whose adjoint gives the desired equivalence $BY_1 \times BY_2 \to BX$.

We start with some preliminary remarks. The maximal torus $T_X \cong T_1 \times T_2$ and the Weyl group $W_X \cong W_1 \times W_2$ split. These splittings give maximal tori and Weyl groups of Y_1 and Y_2. We also have splittings $N_{X_0} \cong N_{1,0} \times N_{2,0}$, $W_{X_0} \cong W_{Y_{1,0}} \times W_{Y_{2,0}}$ and $\pi := \pi_0(X) \cong \pi_1 \times \pi_2$ such that $N_{i,0}$ is the normalizer of T_i in $Y_{i,0}$, such that $W_{i,0} \cong W_{Y_{i,0}}$ and such that $\pi_i = \pi_0(Y_i)$. All this follows from Definition 5.1 and the lemmas 5.2, 5.3 and 5.4. Here $Y_{i,0}$ denotes the component of the unit of Y_i.

Step 3 is now established by the following series of 6 claims and a final argument.

Claim 3.1: The compositions $Y_2 \to X \to X/Y_1$ and $Y_1 \to X \to X/Y_2$ are homotopy equivalences.

Proof. We only discuss the first composition. The triple $(X_0; Y_{1,0}, Y_{2,0})$ satisfies condition (A) by construction. And by hypothesis, this establishes a commutative diagram

$$\begin{array}{ccc}
N_{1,0} \times N_{2,0} & \longrightarrow & N_{X_0} \\
\downarrow & & \downarrow \\
Y_{1,0} \times Y_{2,0} & \xrightarrow{\cong} & X_0 \\
\uparrow & \nearrow & \\
Y_{i,0} & &
\end{array}.$$

Because the lower horizontal arrow is an isomorphism of p–compact groups, the composition $Y_{2,0} \to X_0 \to X_0/Y_{1,0}$ is an equivalence and fits into the commutative diagram of fibrations

$$\begin{array}{ccccc}
Y_{2,0} & \longrightarrow & Y_2 & \longrightarrow & \pi_2 \\
\downarrow \simeq & & \downarrow & & \| \\
X_0/Y_{1,0} & \longrightarrow & X/Y_1 & \longrightarrow & \pi_2 \cong \pi/\pi_1
\end{array},$$

where the middle vertical arrow is given by the composition $Y_2 \to X \to X/Y_1$ and is a homotopy equivalence, too. $\square$

Claim 3.2: If $T_i \to Y_{i,0}$ is central for $i = 1, 2$, then X is a toral p–compact group.

Proof. The Weyl group W_{X_0} acts trivially on T_X by assumption and faithfully by Theorem 2.8. Hence, W_{X_0} is the trivial group, $T_X \to X_0$ is an isomorphism of p–compact groups ([15; 3.7], and X is a p-compact toral group. $\square$

For p-compact toral groups we already proved the statement. Hence, in the following we always can assume that $T_1 \to Y_{1,0}$ is not a central subgroup, that $\mathrm{cd}_{\mathbb{F}_p}(C_{Y_1}(T_1)) < \mathrm{cd}_{\mathbb{F}_p}(Y_1)$, that the Weyl group of $C_{Y_{1,0}}(T_1)$ is a proper subgroup of $W_{Y_{1,0}}$, that the Weyl group of $C_{X_0}(T_1)$ is a proper subgroup of W_{X_0} and that $\mathrm{cd}_{\mathbb{F}_p}(C_{X_0}(T_1)) < \mathrm{cd}_{\mathbb{F}_p}(X_0)$. The first three assertions follow from Lemma 2.17 and the last inequality follows, since otherwise $C_{X_0}(T_1) \cong X_0$ (see [7; 6.14, 6.15] or the proof of Lemma 2.17) and hence $W_{C_{X_0}(T_1)} \cong W_{X_0}$.

Claim 3.3: The composition $BP_1 \times BN_2 \to BN_1 \times BN_2 \to BX$ extends to a map $BP_1 \times BY_2 \to BX$ such that the restriction to the second coordinate is homotopic to the inclusion $BY_2 \to BX$.

Proof. If we make $BT_1 \to BN_1$ into a fibration, we get an W_1–action on BT_1 as well as on $\mathrm{map}(BT_1, BX)_{Bi}$ and on $\mathrm{map}(BT_1, BN_X)_{Bi}$. The components are obviously fixed under this action. The map $BC_{N_X} := BC_{N_X}(T_2) \to BC_X(T_2) =: BC_X$ induces an epimorphism between the fundamental groups (see Lemma 5.4) and so does the map $(BC_{N_X})_{hW_1} \to (BC_X)_{hW_1}$ between the associated Borel constructions. The 5-lemma and Lemma 5.4 (2) and the proof show that $\pi_1((BC_X)_{hW_1}) \cong \pi_1((BC_{N_1})_{hW_1}) \times \pi_1(BY_2)$, where $C_{N_1} := C_{N_1}(T_1)$. Since $\mathrm{map}(BN_1, BN_1) \simeq \mathrm{map}(BT_1, BN_1)^{hW_1}$ (see the proof of Proposition 2.19), the identity $BN_1 \to BN_1$ induces a section of the Borel construction $(BC_{N_1})_{hW_1} \to BW_1$. Therefore, the group $\pi_1((BC_{N_1})_{hW_1}) \cong \pi_1(BC_{N_1}) \rtimes W_1$ splits as a semi direct product.

Now we consider the pull back diagram

$$\begin{array}{ccccc}
BT_1 \times BY_2 & = & BT_1 \times BY_2 & = & BT_1 \times BY_2 \\
\downarrow & & \downarrow & & \downarrow \\
BU_X & \longrightarrow & E & \longrightarrow & BC_X \\
\downarrow & & \downarrow & & \downarrow \\
BS_pW_1 & \longrightarrow & BW_1 & \longrightarrow & B\pi_1((BC_{N_1})_{hW_1})\ ,
\end{array}$$

where $S_pW_1 \subset W_1$ is a p–Sylow subgroup of W_1. The space BU_X is p–complete and can be considered as the classifying space of a p–compact group U. Obviously, we have $\mathrm{cd}_{\mathbb{F}_p}(U) < \mathrm{cd}_{\mathbb{F}_p}(X)$.

We can play the same game with $C_{N_X} \cong C_{N_1} \times N_2$ instead of C_X. This yields a map $BU_{N_X} \to BU_X$, which is the normalizer of a maximal torus of BU_X, and an equivalence $BU_{N_X} \simeq B\widetilde{P} \times BN_2$. The first factor fits into a fibration $BT_1 \to B\widetilde{P} \to BS_pW_1$ and is therefore a p–compact group. There also exist maps $B\widetilde{P} \to BC_{N_1} \to BN_1 \to BY_1$ and this map lifts to BP_1. The lift induces an isomorphism on the fundamental groups and an equivalence between the universal covers which are both given by BT_1. Thus, this lift establishes an isomorphism $\widetilde{P} \cong P_1$ of p–compact groups.

By construction, the map $BY_2 \to BU_X$ is associated to $BN_2 \to BU_X$ as well as $BP \simeq B\widetilde{P} \to BU_X$ to itself. Thus, the triple $(BU_X; BP, BY_2)$ satisfies condition (A). By induction hypothesis, we can assume that there exists an equivalence $BP \times BY_2 \simeq BU_X$ which extends the map $BP \times BN_2 \to BU_X$ and that therefore the desired extension of the claim exists. $\square$

5.6 Remark. The proof of the Claim 3.3 does not depend on the equivalence $Y_1 \simeq X/Y_2$ which we proved in Claim 3.1. We only had to assume that $T_1 \to Y_{1,0}$ is not central.

Claim 3.4: The pull back of the fibration $X/Y_1 \to BY_1 \to BX$ along the composition $BP_1 \to BY_1 \to BX$ is fiber homotopic trivial. Moreover, we have $\mathrm{map}(BP_1, BX)_{Bi} \simeq BZ(P_1) \times BY_2$.

Proof. By Claim 3.1, the space Y_2 can be identified with X/Y_1. Taking the pull back induces a diagram of fibrations

$$(*)\qquad \begin{array}{ccccc} Y_2 & \longrightarrow & E_P & \longrightarrow & BP_1 \\ \| & & \downarrow & & \downarrow \\ Y_2 & \longrightarrow & BY_1 & \longrightarrow & BX \end{array} \quad .$$

Obviously, the induced fibration has a section $s_P : BP_1 \to E_P$ induced by the map $Bj_1 : BP_1 \to BY_1$. Applying the functor $\mathrm{map}(BP_1, \ )$ to the diagram yields a pull back diagram

$$(**)\qquad \begin{array}{ccccc} (Y_2^{hP_1})_{s_P} & \longrightarrow & \mathrm{map}(BP_1, E_P)_{s_P} & \longrightarrow & \mathrm{map}(BP_1, BP_1)_{id} \\ \| & & \downarrow & & \downarrow \\ (Y_2^{hP_1})_{s_P} & \longrightarrow & \mathrm{map}(BP_1, BY_1)_{Bj_1} & \longrightarrow & \mathrm{map}(BP_1, BX)_{Bi_1Bj_1} \end{array} \quad ,$$

where $Y_2^{hP_1}$ denotes the space of sections of the upper fibration in $(*)$ respectively the homotopy fixed–point set of the proxy action given by the upper line in $(*)$ (for definitions see [7; §10]). As fiber we only have to take those components given by sections $BP_1 \to E_P$ which are homotopic to s_P as maps (not as sections).

We show that evaluation at the basepoint induces an equivalence $(Y_2^{hP_1})_{s_P} \to Y_2$. To do this we have to calculate all the other mapping spaces of the diagram $(**)$. The maps $BZ(P_1) \to \mathrm{map}(BP_1, BP_1)_{\mathrm{id}} \to \mathrm{map}(BP_1, BY_1)_{Bj_1}$ are equivalences by Proposition 2.19.

Let $\overline{P}_1 := P_1/T_1$ denote the group of the components of P_1. Making $BT_1 \to BP_1$ into a fibration, i.e. into a covering, we get an action of $\overline{P}_1$ on BT_1 and on $\mathrm{map}(BT_1, \ )$. By [10], we have an equivalence of functors $\mathrm{map}(BP_1, \ ) \simeq \mathrm{map}(BT_1, \ )^{h\overline{P}_1}$. This allows to calculate the mapping space in the lower right corner of the above pull back diagram, which we do next.

By Claim 3.3 there exists an extension $BP_1 \times BY_2 \to BX$ of the map $BP_1 \times BN_2 \to BX$. Applying the functor $\mathrm{map}(BT_1, \ )$ establishes a $\overline{P}_1$–equivariant map $\mathrm{map}(BT_1, BP_1 \times BY_2)_{Bi} \to \mathrm{map}(BT_1, BX)_{Bi}$. (Again all inclusions are denoted by i.) Both mapping spaces are p–compact groups whose component of the unit is given by $T_1 \times Y_{2,0}$. Moreover, the map also induces an isomorphism between the fundamental groups of the spaces (this follows from Lemma 5.4 (2)) and is therefore an equivalence. Passing to homotopy fixed–points establishes equivalences

$$BZ(P_1) \times BY_2 \simeq \mathrm{map}(BP_1, BP_1 \times BY_2)_{Bi} \simeq \mathrm{map}(BP_1, BX)_{Bi} \ .$$

This proves the second part of Claim 3.4.

We used the same trick to calculate the mapping spaces $\mathrm{map}(BP_1, BY_1)_{Bi} \simeq \mathrm{map}(BP_1, BP_1)_{id} \simeq BZ(P_1)$ (Proposition 2.19). Because this trick is functorial, the compositions

$$BZ(P_1) \xrightarrow{\simeq} \mathrm{map}(BP_1, BP_1)_{id} \to \mathrm{map}(BP_1, BX)_{Bi} \simeq BZ(P_1) \times BY_2$$

and

$$BZ(P_1) \xrightarrow{\simeq} \mathrm{map}(BP_1, BY_1)_{Bi} \to \mathrm{map}(BP_1, BX)_{Bi} \simeq BZ(P_1) \times BY_2$$

are given by the inclusion into the first coordinate.

The pull back diagram (∗∗) translates now to the upper half of

$$\begin{array}{ccccc}
(Y_2^{h\overline{P}_1})_{s_P} & \longrightarrow & \mathrm{map}(BP_1, E_P)_{s_P} & \longrightarrow & BZ(P_1) \\
\| & & \downarrow & & \downarrow \\
(Y_2^{h\overline{P}_1})_{s_P} & \longrightarrow & BZ(P_1) & \longrightarrow & BZ(P_1) \times BY_2 \\
\downarrow & & \downarrow & & \downarrow \\
Y_2 & \longrightarrow & BY_1 & \longrightarrow & BX
\end{array} .$$

The maps from the middle row to the bottom row are given by the obvious inclusion or by evaluation at basepoints if we replace the terms in the middle row by mapping spaces. The map $Y_1/Z(P_1) \to X/(Z(P_1) \times Y_2)$ is an equivalence. This follows from Claim 3.1. Hence, the lower right square is a pull back diagram (up to homotopy), and the lower left vertical arrow is a homotopy equivalence. The adjoint of $Y_2 \simeq (Y_2^{h\overline{P}_1})_{s_P} \to \mathrm{map}(BP_1, E_P)_{s_P}$ establishes a homotopy equivalence $BP_1 \times Y_2 \to E_P$ and a trivialization of the fibration $Y_2 \to E_P \to BP_1$. This finishes the proof of Claim 3.4. □

Claim 3.5: The pull back of the fibration $X/Y_1 \to BY_1 \to BX$ along the map $BY_1 \to BX$ is fiber homotopic trivial.

Proof. The fiber $X/Y_1 \simeq Y_2$ (Claim 3.1) is mod–p finite, p–complete and a loop space. The statement now follows directly from Claim 3.4 and Corollary 4.4. □

Claim 3.6: There exists an equivalence $BZ(Y_1) \times BY_2 \xrightarrow{\simeq} \mathrm{map}(BY_1, BX)_{Bi}$ such that the restriction to BY_2 composed with the evaluation at the basepoint is homotopic to the map $BY_2 \to BX$.

Proof. Converting $BY_1 \to BX$ into a fibration, Claim 3.5 establishes a pull back diagram

$$(***) \qquad \begin{array}{ccc}
BY_1 \times Y_2 & \longrightarrow & BY_1 \\
\downarrow & & \downarrow \\
BY_1 & \longrightarrow & BX
\end{array} .$$

Applying the functor map(BY_1,) yields another pull back diagram

$$\begin{array}{ccc}
\mathrm{map}(BY_1, BY_1 \times Y_2)_{Bk} \simeq BZ_1 \times Y_2 & \longrightarrow & \mathrm{map}(BY_1, BY_1)_{id} \simeq BZ_1 \\
\downarrow & & \downarrow \\
\mathrm{map}(BY_1, BY_1)_{id} \simeq BZ_1 & \longrightarrow & \mathrm{map}(BY_1, BX)_{Bi} =: BY_2'
\end{array} ,$$

where $Z_1 := Z(Y_1)$ denotes the center of Y_1 and where k is the inclusion into the first coordinate. The equivalences follow from [8; 1.3] and Theorem 1.4. Passing to loop spaces shows that BY_2' is the classifying space of a p–compact group.

Now we consider the diagram

$$\begin{array}{ccc} BZ_1 \simeq \operatorname{map}(BY_1, BY_1)_{id} & \longrightarrow & BZ(P_1) \simeq \operatorname{map}(BP_1, BY_1)_{Bi} \\ \downarrow & & \downarrow \\ BY_2' \simeq \operatorname{map}(BY_1, BX)_{Bi} & \longrightarrow & BZ(P_1) \times BY_2 \simeq \operatorname{map}(BP_1, BX)_{Bi} \end{array} .$$

Since the restriction of the fibration $BY_1 \to BX$ to BP_1 or BY_1 is fiber homotopic trivial (Claim 3.4 and Claim 3.5), the fiber of the right column is given by the space of section of the trivial fibration $BP_1 \times Y_2 \to BP_1$ and the fiber of the left column by the sections of the trivial fibration $BY_1 \times Y_2 \to BY_1$. Hence, both fibers are homotopy equivalent to $Y_2 \simeq \operatorname{map}(BP_1, Y_2) \simeq \operatorname{map}(BP_1, Y_2)$ and the diagram is a pull back diagram. Because Z_1 is a central subgroup of the terms in the top row and in the lower right corner, an application of the functor $\operatorname{map}(BZ_1, \ )$ shows that $i : Z_1 \subset Y_2'$ is also a central subgroup.

The right column has a section $BZ(P_1) \times BY_2 \to BZ(P_1)$ with homotopy fiber BY_2 and so has the first column. That is to say that there exists a map $BY_2' \to BZ_1$ with homotopy fiber BY_2. Taking the adjoint of $BY_2 \to BY_2' \simeq \operatorname{map}(BZ_1, BY_2')_{Bi}$ gives the desired equivalence $BZ_1 \times BY_2 \simeq BY_2' \simeq \operatorname{map}(BY_1, BX)_{Bi}$. The second part of the claim follows from the construction. □

Now we can finish the proof of Step 3. We can apply the functor $\operatorname{map}(BN_1, \ )$ to the pull back diagram $(***)$. To make all the arguments in the proof of Claim 3.6 working we only have to note that $\operatorname{map}(BN_1, Y_2) \simeq \operatorname{map}(BT_1, Y_2)^{hW_1} \simeq (Y_2)^{hW_1} \simeq map(BW_1, Y_2) \simeq Y_2$. The first equivalence follows if we convert $BT_1 \to BN_1$ into a covering with deck transformation group W_1 (see the proof of Claim 3.4), the second from the Sullivan conjecture for p–compact groups [8] and the last from the Sullivan conjecture for finite groups [13]. We get an equivalence $BZ_1' \times BY_2 \simeq \operatorname{map}(BN_1, BX)_{Bi}$ such that the restriction to BY_2 composed with the evaluation at the basepoint equals the map $BY_2 \to BX$. Here, we set $BZ_1' := \operatorname{map}(BN_1, BY_1)_{Bi} \simeq \operatorname{map}(BN_1, BN_1)_{id}$. In particular, this shows that there exists a commutative diagram

$$\begin{array}{ccc} BN_2 & \longrightarrow & \operatorname{map}(BN_1, BX)_{Bi} \\ \downarrow & & \| \\ BY_2 & \longrightarrow & \operatorname{map}(BN_1, BX)_{Bi} \\ \| & & \uparrow \\ BY_2 & \longrightarrow & \operatorname{map}(BY_1, BX)_{Bi} \end{array} ,$$

where the top row is given by the adjoint of $BN_1 \times BN_2 \simeq BN_X \to BX$. Passing again to adjoints, establishes the commutative diagram

$$\begin{array}{ccc} BN_1 \times BN_2 & \xrightarrow{\cong} & BN_X \\ \downarrow & & \downarrow \\ BY_1 \times BY_2 & \longrightarrow & BX \\ \uparrow & \nearrow & \\ BY_i & & \end{array} .$$

Because the lower horizontal arrow gives a subgroup of maximal rank and induces an isomorphism between the Weyl groups and the groups of the components of the underlying p–compact groups, it establishes an isomorphism of p–compact groups (see the proof of Lemma 2.17). This completes the proof of the third step.

Step 4: Let X be a connected p–compact group, let $Z := Z(X)$ denote the center of X and $\overline{X} := X/Z$ the associated centerfree p–compact group. We have to show that $X \in \mathcal{C}l$ if $\overline{X} \in \mathcal{C}l$.

Let $N_X \cong N_1 \times N_2$, let $Y_1, Y_2 \to X$ be the associated subgroups, let $P_i \subset N_i \subset Y_i$ be a Sylow p–toral subgroup of Y_i, and let $Z_i := Z(Y_i)$ denote the center of Y_i. The center Z_i is a subgroup of P_i [15; 4.3]. Because W_i acts trivially on Z_i, the inclusion $Z_i \subset N_i$ is central [17; 3.7]. By [2; 7.2], we can construct a diagram of principal bundles

$$\begin{array}{ccccc}
BZ_1 \times BZ_2 & \longrightarrow & BN_1 \times BN_2 & \longrightarrow & B\overline{N}_1 \times B\overline{N}_2 \\
\| & & \downarrow & & \downarrow \\
BZ & \longrightarrow & BX & \longrightarrow & B\overline{X} \\
\uparrow & & \uparrow & & \uparrow \\
BZ_i & \longrightarrow & BY_i & \longrightarrow & B\overline{Y}_i
\end{array} .$$

By construction, the subgroup $\overline{Y}_i \to \overline{X}$ is associated to $\overline{N}_i$. Thus, by hypothesis, there exists an equivalence $B\overline{Y}_1 \times B\overline{Y}_2 \to B\overline{X}$ extending the map on the normalizers. The diagram of the classifying maps of the principle bundles

$$\begin{array}{ccc}
B\overline{Y}_1 \times B\overline{Y}_2 & \longrightarrow & B^2Z_1 \times B^2Z_2 \\
\downarrow \simeq & & \| \\
B\overline{X} & \longrightarrow & B^2Z
\end{array}$$

commutes up to homotopy. This follows since homotopy classes of the horizontal arrows are determined by cohomology classes and since the restrictions of both maps to the normalizers are equal. The identity in the right column follows from Lemma 5.4. The map between the fibers gives the desired equivalence $BY_1 \times BY_2 \to BX$. This maps is obviously an extension of $BN_1 \times BN_2 \to BX$ and fits together with the maps $BY_i \to BX$. This finishes the proof of Step 4.

Step 5: Let X be a centerfree p–compact group. We have to show that the statement is true for X, if it is satisfied by all p–compact groups with smaller cohomological dimension. We prove this as in Step 3. Let X be a centerfree p–compact group and $(X; Y_1, Y_2)$ a triple satisfying condition (A). As an inspection of the arguments shows we only have to prove Claim 3.1 of Step 3, namely that the compositions $Y_1 \to X \to X/Y_2$ and $Y_2 \to X \to X/Y_1$ are homotopy equivalences. This turns out to be quite complicated and is done in several parts.

The outline is as follows: The first major step is the construction of an equivalence between the Quillen categories of X and $Y_1 \times Y_2$ (Claim 5.2). Let F_X :

$A_p(X) \to \mathcal{A}b$ be the functor given by $F_X(E \to X) := H^*(BC(E); \mathbb{F}_p)$. Analogously, we define a functor $F_Y : A_p(Y_1 \times Y_2) \to \mathcal{A}b$. Using the equivalence of the Quillen categories, we then show that there exists a natural transformation $F_X \to F_Y$, which is an isomorphism on the objects (Claim 5.5). The mod-p decomposition theorem of p–compact groups (Theorem 2.15) establishes an isomorphism $H^*(BX; \mathbb{F}_p) \cong H^*(BY_1 \times BY_2; \mathbb{F}_p)$ which is compatible with the equivalence $BN_X \simeq BN_1 \times BN_2$ (Claim 5.6). Then, an Eilenberg–Moore spectral sequence argument allows to calculate the homotopy fibers of $BY_1 \to BX$ and $BY_2 \to BX$ which turn out to be equivalent to Y_2 and Y_1. (Claim 5.7).

Claim 5.1: Let $j_1, j_2 : E \to P := P_1 \times P_2$ be two monomorphisms of an elementary abelian subgroup E into P and let $i_X : P \to X$ be the obvious inclusion. Then, $Bi_Y Bj_1 \simeq Bi_Y Bj_2$ if and only if $Bi_X Bj_1 \simeq Bi_X Bj_2$.

Proof. We prove the statement via an induction over the rank of E. Let us assume that $E = \mathbb{Z}/p$. Since X is centerfree and since $Z \cong Z(Y_1) \times Z(Y_2)$ (Lemma 5.4), the monomorphisms $T_i \to Y_i$ are not central. Hence, by Remark 5.6, there exists a map $BP_1 \times BY_2 \to BX$ extending $BP_1 \times BP_2 \to BX$. And both compositions $Bi_Y Bj_i$ and $Bi_X Bj_i$ factor over $BP_1 \times BY_2$. Because Y_2 is connected (Lemma 5.4), the second coordinate of both maps, as a cyclic subgroup, is subconjugate to T_2 (Theorem 2.6). Hence, we can assume that the second coordinate of $j_i : E \to P_1 \times P_2$ takes image in T_2. Analogously, we can assume that the first coordinate also takes image in T_1. That is to say that $j_i : E \to P_1 \times P_2$ represents E as a subgroup of the torus. By [16; 4.2], the two maps $Bi_Y Bj_1$ and $Bi_Y Bj_2$ are homotopic if and only if Bj_1 and Bj_2 differ by a Weyl group element if and only if $Bi_X Bj_1 \simeq Bi_X Bj_2$. This proves Claim 5.1 for $E = \mathbb{Z}/p$.

Now let us assume that the rank of E is bigger than 1. Let $E_1 \subset E$ denote the first coordinate of E and $E_2 \subset E$ a complement of E_1, i.e. $E \cong E_1 \times E_2$. By what we already proved we can assume that, for $i = 1, 2$, the restrictions $j_1|_{E_1} = j_2|_{E_1} : E_1 \to P_1 \times P_2$ are subgroups of the maximal torus. Let $Bj_i' : BE_2 \to BC_{P_1 \times P_2}(E_1)$ be the adjoint of Bj_i. The centralizers $X' := C_X(E_1)$ and $C_{Y_1 \times Y_2}(E_1)$ are subgroups of maximal rank and have smaller cohomological dimension than the centerfree group X. The normalizer $N_{C_X(E_1)} \cong C_{N_X}(E_1) = C_{N_1}(E_1) \times C_{N_2}(E_1)$ of a maximal torus of $C_X(E_1)$ splits again, and $Y_i' := C_{Y_i}(i_{Y_i} j_i(E_1)) \to Y_i$ is the subgroup associated to $N_i' := C_{N_i}(E_1)$. Therefore, by induction hypothesis, there exists an isomorphism $Y_1' \times Y_2' \xrightarrow{\simeq} X'$ of p–compact groups making the diagram

$$\begin{array}{ccc} & BN_1' \times BN_2' & \\ \swarrow & & \searrow \\ BY_1' \times BY_2' & \longrightarrow & BX' \end{array}$$

commutative up to homotopy.

The two maps $Bi_X Bj_1, Bi_X Bj_2 : BE_1 \times BE_2 \to BX$ are homotopic if the adjoints $BE_2 \to BX'$ are homotopic. The same is true if we replace X by $Y_1 \times Y_2$, X' by $Y_1' \times Y_2'$ and Bi_X by Bi_Y. The above homotopy commutative diagram shows that the claim is true for the adjoints. This finishes the induction and the proof of Claim 5.1.

Claim 5.2: The Quillen categories of X and $Y_1 \times Y_2$ are isomorphic, i.e. there exists a functor $\mathcal{A}_p(Y_1 \times Y_2) \to \mathcal{A}_p(X)$, which is an isomorphism of categories.

Proof. By Proposition 2.10, every elementary abelian subgroup is subconjugate to $P_1 \times P_2$. Hence the statement follows from Claim 5.1.

Claim 5.3: Let $j : E \to P_X := P_1 \times P_2$ be a monomorphism. Then, the map j is a special subconjugation for $i_Y j$ if and only of it is a special subconjugation for $i_X j$.

Proof. Let $E \xrightarrow{j} P_X$ be a special subconjugation of $i_Y j$ and $E \xrightarrow{j'} P_X$ a special subconjugation of $i_X j$. By Claim 5.1, the composition $i_Y j'$ is conjugate to $i_Y j$. Hence, the centralizer $P' = C_{P_X}(j'(E))$, which is the Sylow p–toral subgroup of $C_X(i_X j(E))$, is subconjugate to the Sylow p–toral subgroup $P = C_{P_X}(j(E))$ of $C_Y(i_Y j(E))$ and vice versa. Therefore P and P' are isomorphic. This proves the statement. □

Claim 5.4: Let $E \xrightarrow{j} P_X$ be a monomorphism and a special subconjugation of $i_X j$ and $i_Y j$. Then there exists an equivalence $f_E : BC_Y(E) \xrightarrow{\simeq} BC_X(E)$ making the diagram

$$\begin{array}{ccccc} & & BC_{P_X}(E) & & \\ & \swarrow & & \searrow & \\ BC_Y(E) & & \xrightarrow[f_E]{\simeq} & & BC_X(E) \end{array}$$

commutative up to homotopy.

Proof. Let $E_1 \cong \mathbb{Z}/p \subset E$ denote the first coordinate of E. By the proof of Claim 5.1 and Lemma 2.14 we can assume that $E_1 \to P_X$ is a toral subgroup and a special subconjugation of $E_1 \to P_X \to X$ and $E_1 \to P_X \to Y$. As shown in the proof of Claim 5.1 (using the induction hypothesis), there exists a homotopy commutative diagram

$$\begin{array}{ccccc} & & BC_{P_X}(E_1) & & \\ & \swarrow & & \searrow & \\ BC_Y(E_1) & & \xrightarrow[f_{E_1}]{\simeq} & & BC_X(E_1) \end{array} .$$

Passing to adjoints and taking centralizers finishes the proof of Claim 5.4. □

Claim 5.5: There exists a natural transformation $F_Y \to F_X$, which is an isomorphism on objects.

Proof. Let $E \xrightarrow{j} P_X$ be a special subconjugation of $i_Y j$ and $i_X j$. Then we define

$$\phi_E := H^*(f_E, \mathbb{F}_p) : H^*(BC_Y(E); \mathbb{F}_p) \to H^*(BC_X(E); \mathbb{F}_p) .$$

Let $E_1 \to E_2$ be a morphism of $A_p(Y) \cong A_p(X)$. By Lemma 2.14, for $i = 1, 2$, there exist monomorphisms $j_i : E_i \to P_X$, which are special lifts of $i_Y j_i$ and $i_X j_i$

such that the diagram

$$\begin{array}{ccc} BE_1 & \longrightarrow & BE_2 \\ & {\scriptstyle j_1}\searrow \qquad \swarrow{\scriptstyle j_2} & \\ & BP_X & \end{array}$$

commutes up to homotopy. Passing to centralizers gives another diagram, namely

$$\begin{array}{ccccc} & BC_Y(E_2) & \longrightarrow & BC_Y(E_1) & \\ \nearrow & & \nearrow & & \\ BC_{P_X}(E_2) & \longrightarrow \ \downarrow {\scriptstyle f_{E_2}} & BC_{P_X}(E_1) & \downarrow {\scriptstyle f_{E_1}} & \\ \searrow & & \searrow & & \\ & BC_X(E_2) & \longrightarrow & BC_X(E_1) & . \end{array}$$

The two triangles commute up to homotopy (Claim 5.4) as well as the upper and the lower parallelogram. For any p–compact group U with Sylow p–toral subgroup $P_U \to U$, the map $H^*(BU;\mathbb{F}_p) \to H^*(BP_U;\mathbb{F}_p)$ is an injection (Proposition 2.10). The diagonal arrows induce injections in mod-p cohomology, because we always chose special subconjugation. Therefore, the square commutes at least in mod-p cohomology, and the maps Φ_E establish a natural transformation $F_Y \to F_X$, which is an isomorphism on the values of the objects. □

Claim 5.6: There exists an isomorphism

$$\phi : H^*(BX;\mathbb{F}_p) \xrightarrow{\cong} H^*(BY;\mathbb{F}_p)$$

of algebras over the Steenrod algebra such that the composition $H^*(BX;\mathbb{F}_p) \xrightarrow{\phi} H^*(BY;\mathbb{F}_p) \to H^*(BY_i,\mathbb{F}_p)$ equals the map $H^*(BX;\mathbb{F}_p) \to H^*(BY_i;\mathbb{F}_p)$ induced by the maps $BY_i \to BX$.

Proof. We have the following sequence of isomorphisms:

$$H^*(BX;\mathbb{F}_p) \cong \varprojlim_{\mathcal{A}_p(X)} F_X \cong \varprojlim_{\mathcal{A}_p(Y)} F_Y \cong H^*(BY;\mathbb{F}_p)\ .$$

The first and last isomorphism follow from Theorem 2.15 and the middle isomorphism from Claim 5.5. The second part of the statement follows from the construction of the natural transformation $F_Y \to F_X$. □

Claim 5.7: The compositions $Y_2 \to X \to X/Y_1$ and $Y_1 \to X \to X/Y_1$ are homotopy equivalences.

Proof. In the pull back diagram

$$\begin{array}{ccc} E & \longrightarrow & BY_1 \\ \downarrow & & \downarrow \\ BY_2 & \longrightarrow & BX \end{array}$$

we calculate the mod–p cohomology of the space E via the Eilenberg–Moore spectral sequence. The E_2–term is given by

$$\begin{aligned} E_2 =& Tor_{H^*(BX;\mathbb{F}_p)}(H^*(BY_1;\mathbb{F}_p), H^*(BY_2;\mathbb{F}_p)) \\ \cong& Tor_{H^*(BY_1;\mathbb{F}_p)\otimes H^*(BY_2;\mathbb{F}_p)}(H^*(BY_1;\mathbb{F}_p), H^*(BY_2;\mathbb{F}_p) \\ \cong& Tor_{H^*(BY_1;\mathbb{F}_p)}(H^*(BY_1;\mathbb{F}_p), \mathbb{F}_p) \\ \cong& Tor_{\mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p) \ . \end{aligned}$$

The functor $\otimes_{\mathbb{F}_p} H^*(BY_i;\mathbb{F}_p)$ is exact. This implies the third isomorphism and also, together with Claim 5.6, the second isomorphism. Therefore, the space E is mod–p acyclic. Moreover, E is p–complete, because all the other involved spaces are p–complete, Hence, $E \simeq *$. The map between the fibers of the columns are obviously given by the composition $Y_2 \to X \to X/Y_1$, which is a homotopy equivalence. Looking at the fibers of the rows establishes the equivalence $Y_1 \to X \to X/Y_2$. This proves the statement. □

Having shown the equivalences $Y_2 \to X/Y_1$ and $Y_1 \to X/Y_2$ we can proceed as in Step 3. This finishes the proof of the last step of the induction as well as the proof of Theorem 5.5. □

6. Proof of Theorem 1.1 and Theorem 1.2.

In the following a finite extension N of a p-compact torus T is called *simple* if the associated representation $N/T \to Gl(H_2(BT;\mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is irreducible and faithful.

6.1 Lemma. *Let N be a finite extension of a p-compact torus T, such that N/T acts faithfully as a pseudo reflection group on $H_2(BT;\mathbb{Z}_p^\wedge)\otimes\mathbb{Q}$. Then, up to order and isomorphisms, there exists at most one splitting $N \cong N_1 \times ... \times N_n$ into simple finite extensions of p–compact tori.*

Proof. The existence of a splitting into simple factors shows that we have splittings $W := N/T \cong W_1 \times ... \times W_n$ for the quotient, $L_T := H_2(BT;\mathbb{Z}_p^\wedge) \cong L_1 \oplus ... \oplus L_n$ for the 2–dimensional homology of BT and $T \cong T_1 \times ... \times T_n$ for the torus T itself (see Lemma 5.2). By standard representation theory of pseudo reflection groups, these splittings are unique up to order. Thus, the splitting of N is also unique up to order. □

Before we start with the construction of splittings we first prove the uniqueness statement of Theorem 1.1 and Theorem 1.2. This is contained in the next statement.

6.2 Theorem. *Let $X_1 \times ... \times X_n \cong X \cong Y_1 \times ... \times Y_m$ be splittings of a connected p–compact group X into products of simple connected p–compact groups. Then, $m = n$ and there exists a permutation $\sigma \in \Sigma_n$ and isomorphisms $\phi_i : X_i \xrightarrow{\cong} Y_{\sigma(i)}$ such that the diagram*

$$\begin{array}{ccc} & X & \\ \swarrow^{\cong} & & \searrow^{\cong} \\ \prod_i X_i & \xrightarrow{\sigma \prod_i \phi_i} & \prod_i Y_i \end{array}$$

commutes up to conjugation.

Proof. Let $T_X \to X$ be a maximal torus of X and $N_X \to X$ the normalizer of $T_X \to X$. Each splitting of X into simple pieces gives rise to a splitting of N_X

into simple factors. By Theorem 2.8, the normalizer N_X satisfies the assumption of Lemma 6.1. Hence, there exist splittings $T_X \cong T_1 \times ... \times T_n$ and $N_X \cong N_1 \times ... \times N_n$ into simple factors unique up to order and isomorphisms. In particular, this implies that $n = m$, that there exists a permutation $\sigma \in \Sigma_n$ such that $X_i \to X$ and $Y_{\sigma(i)} \to X$ are subgroups associated to $N_i \to N_X \to X$ and that there exists a diagram of loop spaces

$$\begin{array}{ccc} & N_X \cong N_1 \times ... \times N_n & \\ \swarrow & & \searrow \\ X_1 \times ... \times X_n & \xrightarrow[\cong]{\phi} & Y_{\sigma(1)} \times ... \times Y_{\sigma(n)} \end{array}$$

commutative up to conjugation. The diagonal arrows are given by the product of the inclusions $N_i \to X_i, Y_{\sigma(i)}$ and the map ϕ is given by the composition of the two splittings of X. Reordering the factors Y_i we can assume that σ is the trivial permutation.

Let $\phi_{k,l} : X_l \to \prod_i X_i \xrightarrow{\phi} \prod_i Y_i \to Y_k$ be the composition of the inclusion of the l-th factor, of ϕ and of the projection on the k-th factor. For $k \neq l$ the restriction of $B\phi_{k,l}$ to BN_l as well as to BT_l is null homotopic. Because $T_l \to X_l$ is a maximal torus, this implies that $B\phi_{k,l}$ is null homotopic [14; 5.7]. The homomorphisms $\phi_{k,k} : X_k \to Y_k$ induces isomorphisms between the Weyl groups and are therefore isomorphisms between connected p–compact groups [15; 3.7]. Lemma 2.18 now implies that $B\phi \simeq \prod_k B\phi_{k,k}$, which finishes the proof of the statement. □

Finally we prove the existence of the desired splittings. For centerfree connected p–compact groups (Theorem 1.2) this is contained in the next theorem and for simply connected p–compact groups (Theorem 1.1) in Theorem 6.4.

6.3 Theorem. *Let p be an odd prime. Let X be a centerfree connected p–compact group. Then, the following holds:*

(1) *The normalizer $N_X \cong N_1 \times ... \times N_n$ splits into a product of simple factors. For each factor, the quotient $W_i := N_i/T_i$ acts on $H_2(BT_i; \mathbb{Z}_p^\wedge)$ as a pseudo reflection group.*
(2) *The p–compact group $X \cong X_1 \times ... \times X_n$ splits into a product of simple centerfree connected p–compact groups such that the inclusions $X_i \to X$ are associated to $N_i \to N_X \to X$. Moreover, the diagram*

$$\begin{array}{ccc} \prod BN_i \simeq BN_X & \longrightarrow & BN_X \\ \downarrow & & \downarrow \\ \prod_i BX_i & \longrightarrow & BX \end{array}$$

commutes up to homotopy.

Proof. The action of W_X on $L := L_{T_X} = H_2(BT_X; \mathbb{Z}_p^\wedge)$ represents W_X as a pseudo reflection group. By assumption X is centerfree and p is odd. By [8; §7], the W_X–lattice $L := L_X := H_2(BT_X; \mathbb{Z}_p^\wedge)$ is centerfree in the sense of [18; 1.1], and by [18; 1.3] the lattice $L \cong L_1 \oplus ... \oplus L_n$ splits into a direct sum of sublattices such that W_i acts trivially on L_j for $i \neq j$ and such that $L_i \otimes \mathbb{Q}$ is an irreducible W_i

representation. Moreover, there exists a splitting $T := T_X \cong T_1 \times ... \times T_n$ such that $L_i \cong H_2(BT_i; \mathbb{Z}_p^\wedge)$ is an isomorphism of W_X–modules (see [18; §1]).

The fibration $BT_X \to BN_X \to BW_X$ is classified by a map $c_N : BW_X \simeq \prod_i BW_i \to BHE(BT_X)$. Let $c_0 : BW_X \to \prod_i BHE(BT_i) \to BHE(BT_X)$ be the classifying map of the product of the fibrations $BT_i \to BN' \to BW_i$ given by the Borel construction of an action of W_i on BT_i. Since

$$\prod_i BSHE(BT_i) \simeq \prod_i B^2T_i \simeq B^2T_X \simeq BSHE(BT_X),$$

(see the proof of Lemma 4.6) the difference between c_N and c_0 is measured by a cohomology class in

$$H^3(BW_X; \prod_i \pi_3(B^2T_i)) \cong H^3(BW_X; \prod L_i) \cong \prod_i H^3(BW_i; L_i) \ .$$

The last isomorphism follows from Lemma 2.2. Hence, the map c_N also describes a product of fibrations of the desired form which is fiber homotopy equivalent to $BT_X \to BN_X \to BW_X$. This establishes a splitting into simple factors and proves (1).

Now we can apply Theorem 1.3. Via an induction this proves part (2). By Lemma 5.3, each factor is centerfree. □

6.4 Theorem. *Let p be an odd prime. Let X be a simply connected p–compact group. Then, the following holds:*

(1) *The normalizer $N_X \cong N_1 \times ... \times N_n$ splits into a product of simple factors. For each factor, the quotient $W_i := N_i/T_i$ acts on $H_2(BT_i; \mathbb{Z}_p^\wedge)$ as a pseudo reflection group.*

(2) *The p–compact group $X \cong X_1 \times ... \times X_n$ splits into a product of simple simply connected p–compact groups such that the inclusions $X_i \to X$ are associated to $N_i \to N_X \to X$. Moreover, the diagram*

$$\begin{array}{ccc} \prod_i BN_i \simeq BN_X & \longrightarrow & BN_X \\ \downarrow & & \downarrow \\ \prod_i BX_i & \longrightarrow & BX \end{array}$$

commutes up to homotopy.

Proof. Let $\overline{X} := X/Z(X)$ be the associated centerfree connected p–compact group. The center $Z := Z(X)$ of X is a finite group [15; 5.3]. A maximal torus of $\overline{X}$ is given by the inclusion $T_X/Z(X) \to \overline{X}$ and the associated normalizer is given by $N_{\overline{X}} \cong N_X/Z(X) \to \overline{X}$. All these spaces fit into a diagram of principal fibrations

$$\begin{array}{ccccc} BZ & \longrightarrow & BN_X & \longrightarrow & BN_{\overline{X}} \\ \| & & \downarrow & & \downarrow \\ BZ & \longrightarrow & BX & \longrightarrow & B\overline{X} \end{array} \quad .$$

with classifying maps $BN_{\overline{X}} \to B\overline{X} \to B^2Z$ [2; 7.2]. The map $B\overline{X} \to B^2Z$ induces an isomorphism $\pi_2(B\overline{X}) \cong \pi_1(BZ) \cong Z$.

By Theorem 6.3, there exist splittings $N_{\overline{X}} \cong \prod_i \overline{N}_i$ of $N_{\overline{X}}$ and $\overline{X} \cong \prod_i \overline{X}_i$ of $\overline{X}$ into simple centerfree factors such that $\overline{X}_i$ is associated to $\overline{N}_i$. Let $Z \cong \prod_i Z_i$ be the splitting given by the isomorphisms $Z_i \cong \pi_2(B\overline{X}_i)$. Then, the classifying map $B\overline{X} \to B^2Z(X)$ also splits into a product of maps $B\overline{X}_i \to B^2Z_i$. The product of the fibers X_i gives a splitting of $X \cong \prod_i X_i$ into simple simply connected p–compact groups. This also establishes a splitting of $N_X \cong \prod_i N_{X_i}$ into simple factors. This proves the first half of part (2) and part (1). The commutativity of the diagram follows from Theorem 6.3 and from the construction. This finishes the proof of the theorem. □

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