CLASSIFYING SPACES OF COMPACT LIE GROUPS AND FINITE LOOP SPACES

D. Notbohm

The basic problem of homotopy theory is to classify spaces and maps between spaces up to homotopy by means of algebraic invariants such as homology or cohomology. Since their discovery, classifying spaces of compact Lie groups G, denoted by BG, have been a very important part in homotopy theory. For example, they appeared as target in the set of homotopy classes of maps $[X,Y]$, because of their central role in bundle theory. In the last decade, some striking progress was made in the understanding of the homotopy theory of classifying spaces of compact Lie groups. We mention some aspects:

It has been shown that, for a simple connected compact Lie group G, two self maps of BG are homotopic if and only if they induce the same map in rational cohomology.

It also has been proved that for a large class of simply connected compact Lie groups G the mod-p cohomology with cup products and Steenrod operations completely determines the homotopy type of the p–adic completion $BG_p^\wedge$ of BG (for odd primes this contains all classical matrix groups).

Similar methods have also been used to obtain new results for Steenrod's problem: which polynomial algebras can be realized as the mod–p cohomology of a space?

The program of understanding 'classical' Lie group theory from the homotopy point of view, i.e. to express Lie group theory in terms of classifying spaces, is developed to a large extent and might lead to a complete classification of finite loop spaces.

The study of maps between classifying spaces goes back to Hurewicz. For aspherical spaces X and Y he showed that

$$[X,Y]\to Hom(\pi_1(X),\pi_1(Y))/Inn(\pi_1(Y))$$

is a bijection. In particular this applies to classifying spaces of finite or more generally of discrete groups. Here, $Hom(\ ,\ )$ denotes the set of homomorphisms between groups and $Inn(\ )$ the group of inner automorphisms. Moreover, the homotopy type of an aspherical space is determined by the fundamental group. This fed the hope that, up to homotopy, every map between the classifying spaces of any pair of compact Lie groups is induced by a homomorphism. However, in 1970, the first counterexamples were constructed by Sullivan, namely self maps of $BU(n)$, which even in rational cohomology do not look like a map coming from a homomorphism. Inspired by the Sullivan's work, a careful investigation of Hubbuck, Mahmud and

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Adams gave necessary criteria for the effect that maps between classifying spaces of compact Lie groups could have in rational cohomology.

The idea of developing Lie group theory in terms of homotopy theory goes back to Rector. In his study of loop structures on S^3 and sub–loop spaces of finite loop spaces first definitions of basic notions of Lie group theory appeared in terms of classifying spaces, such as homomorphisms, subgroups, maximal tori and Weyl groups.

The proof of the Sullivan conjecture by Miller and Carlsson and subsequent work of Lannes was the break through for the recent fast development in this area. The Sullivan conjecture states as follows:

0.1 Theorem. *(Sullivan conjecture)* [Mil] *Let π be a locally finite group and K a finite CW-complex. Then, the evaluation at a basepoint*

$$map(B\pi, K) \to K$$

is a weak equivalence.

Lannes developed machinery for a purely algebraic calculation of the mod-p cohomology of mapping spaces of the form $map(B\mathbb{Z}/p, X)$. Under some mild assumptions his T–functor calculates $H^*(map(B\mathbb{Z}/p, K); \mathbb{F}_p)$ as an algebra over the Steenrod algebra only using the mod-p cohomology $H^*(X, \mathbb{F}_p)$ as input. For example, this led to a complete description of the mapping space $map(BP, BG)$ for any p–toral groups P and any compact Lie group G, due to Dwyer and Zabrodsky [D–Z] and the author [No 2].

Based on this and a decomposition of BG into a homotopy direct limit of classifying spaces of certain p–toral groups [J–M–O 1], Jackowski, McClure and Oliver set up a program for studying maps $BG \to BH$ for any pair of compact Lie groups G and H. In the case of $G = H$ being a simple connected compact Lie group the program went through and led to:

0.2 Theorem. [J–M–O 1] *Let G be a simple connected compact Lie group. Then two self maps $f, g : BG \to BG$ are homotopic if and only if $f^* = g^* : H^*(BG; \mathbb{Q}) \to H^*(BG; \mathbb{Q})$.*

Lannes' theory and the Jackowski–McClure–Oliver approach also allowed the homotopy type of the classifying space for a large class of compact Lie groups to be characterized.

0.3 Theorem. [No 5] *Let p be an odd prime. Let G be a simply connected compact Lie group such that $H^*(G; \mathbb{Z})$ has no p–torsion, and let X be a space. Then, the p–adic completion $X_p^\wedge$ and $BG_p^\wedge$ are homotopy equivalent if and only if $H^*(X; \mathbb{F}_p)$ and $H^*(BG; \mathbb{F}_p)$ are isomorphic as algebras over the Steenrod algebra.*

The same result for $G = SU(2)$ and $G = SO(3)$ was proved by Dwyer, Miller and Wilkerson for all primes [D–M–W 1], which was the first homotopy uniqueness theorem. The same authors also proved Theorem 1.3 for primes not dividing the order of the Weyl group W_G of G without any extra assumption on G beside being connected [D–M–W 2].

From the homotopy point of view the essential property of a compact Lie group G is the existence of a classifying space BG and a finiteness condition on G, namely that G is a finite CW–complex or a little weaker that $H^*(G; \mathbb{Z})$ is a finitely generated module over $\mathbb{Z}$. Because completion always makes life easier in homotopy theory, these facts led Dwyer and Wilkerson to the definition of p–compact groups. For p–compact groups, the classifying space has to be p–complete, and the finiteness condition is expressed in terms of mod–p cohomology. The main examples are given by completing a connected compact Lie group and the associated classifying space. A generalization of Smith–theory to actions of finite p-groups on $\mathbb{F}_p$–finite p–complete spaces allowed Dwyer and Wilkerson to achieve the following fundamental result in the Lie group theory of p–compact groups. It generalizes well known facts about compact Lie groups. Here, a space is called $\mathbb{F}_p$–finite if the mod–p cohomology is finite.

0.4 Theorem. [D–W 5]

(1) *For any p–compact group X, there exists a maximal torus T_X of X and a Weyl group W_X.*
(2) *If X is connected, the inclusion of the maximal torus induces an isomorphism $H^*(BX;\mathbb{Z}_p^\wedge)\otimes\mathbb{Q} \cong (H^*(BT_X;\mathbb{Z})\otimes\mathbb{Q})^{W_X}$. The representation $W_X \to Gl(H^2(BT_X;\mathbb{Z})\otimes\mathbb{Q})$ is faithful and represents W_X as a pseudo reflection group.*

We think, that these are some of the high lights of the recent achievements in the study of classifying spaces. The idea of this article is to describe developments after the proof of the Sullivan conjecture.

We strongly encourage the reader to take a look at the very nice survey article of Jackowski, McClure and Oliver [J–M–O 4] on a very similar topic. Parts of this article are covered in their paper, in particular the discussion about decomposition and maps between classifying spaces of compact Lie groups. Because this is needed for an understanding of the later development, and because we like to keep this article self contained, we also present this part of the homotopy theory of classifying spaces of compact Lie groups, but much more briefly. For example we omit completely the discussion about the computation of higher inverse limits.

At the end we add as appendix some remarks and facts about homotopy colimits, Lannes' theory and Smith–theory for homotopy fixed–points, which we feel is necessary for an understanding of this article by non experts.

1. Decompositions of classifying spaces.

In the analysis of maps between classifying spaces, decompositions into simpler pieces have proved to be quite a powerful tool. By simpler pieces, we mean classifying spaces of subgroups. There are two different types of such decompositions. One uses centralizers of elementary abelian subgroups. The other is based on p–toral subgroups. Both types are useful for different problems as we will show later.

The idea of decompositions or approximations of classifying spaces goes back to Adams [Ad]. In his analysis of the effect, which maps between classifying spaces may have in complex K–theory, he approximated p–toral subgroups by their finite p–subgroups. This construction was extended by Feshbach to the case of finite extensions of tori [Fe]. For any such extension N, he showed that there exists a sequence

$$M_1 \subset M_2 \subset M_3 \subset .... \subset N$$

of finite subgroups such that

$$\operatorname*{hocolim}_{\mathbb{N}} BM_i \simeq Tel(BM_i) \simeq BM_\infty \to BN$$

is a homology equivalence for any kind of coefficients taken in a finite group. Here, the group $M_\infty := \bigcup_i M_i$ is a locally finite group and gives an approximation of BN at any prime p.

This approach was extended further by Friedlander and Mislin [F–M1,2]. For a compact Lie group G they showed the existence of a locally finite group with similar properties. A locally finite group is the union of finite groups.

1.1 Theorem. *(*[F–M1,2]*) Let G be a compact Lie group, and let q be a prime not dividing the order of $\pi_0(G)$. Then there exists a locally finite group γ and a map $B\gamma \to BG$ which is a mod–p equivalence for any prime p different from q.*

In general, the finite subgroups of γ may not be subgroups of G and the restriction of the map to the classifying space of a finite subgroup may not be induced by a homomorphism. For $G = U(n)$, and a prime p the approximation is given by a map $BGL(n, \overline{\mathbb{F}}_q) \to BU(n)$, where $(q,p) = 1$ and where $\overline{\mathbb{F}}_q$ is the algebraic closure of the field $\mathbb{F}_q$ of q elements.

As a consequence of this approximation theorem, Friedlander and Mislin generalized the Sullivan conjecture to the case of compact Lie groups. For a $\mathbb{F}_p$–finite p–complete space X, they showed that the evaluation at a basepoint $map(BG, X) \to X$ is an equivalence [F–M 2]. The approximation was also used by Mislin to get a complete classification of self maps of $BSU(2)$ up to homotopy [Mis]. This was the first case of such analysis beyond the "simple" case of finite groups or tori.

Although decompositions via centralizers of elementary abelian subgroups or via p–toral subgroups seem to be more useful in the study of maps between classifying spaces, it would be of great interest to have an analogue of Theorem 1.1 for p–compact groups. Examples of this form are given by calculations of Quillen on the cohomology of general linear groups of finite fields [Qu 2].

Decompositions other than telescope constructions were first introduced by Dwyer, Miller and Wilkerson. The pushout of the diagrams

$$(1) \qquad \begin{array}{ccc} SO(3)/D(8) & \longrightarrow & SO(3)/O(2) \\ \downarrow & & \\ SO(3)/\Sigma_4 & & \end{array} \qquad\qquad \begin{array}{ccc} BD(8) & \longrightarrow & BO(2) \\ \downarrow & & \\ B\Sigma_4 & & \end{array}$$

is $\mathbb{F}_2$–acyclic for the left side, and mod–2 equivalent to $BSO(3)$ for the right side. For the left side, this is not to hard to check by explicit calculations, and for the right side, it follows because the Borel construction $EG \times_G -$, as a homotopy colimit, commutes with pushouts. Here, $D(8)$ denotes the dihedral subgroup of $SO(3)$, $O(2)$ the normalizer of the maximal torus and Σ_4 the octohedral subgroup.

There exists a closely related decomposition of $BSO(3)$ at the prime 2, given by the diagram

$$(2) \qquad \Sigma_3 \circlearrowright ESO(3)/(\mathbb{Z}/2)^2 \overset{\longrightarrow}{\rightrightarrows} ESO(3)/O(2)\ .$$

That is that the underlying category has two objects 0 and 1 and the morphisms sets are given by $End(0) = \Sigma_3$, $End(1) = \{id\}$, $Hom(0,1) = \Sigma_3/\Sigma_2$ and $Hom(1,0) = \emptyset$. Decomposition in this case means that of th right diagram is mod 2 equivalent to $BSO(3)$. Notice that the left space is equivalent to $B(\mathbb{Z}/2)^2$and the right to $BO(2)$. A cohomological calculation based on the spectral sequence of Theorem A.1 gives a proof. It also can be shown directly that the homotopy colimit of diagram (2) is equivalent to the pushout of diagram (1).

Unlike the pushout diagram, the decomposition of $BSO(3)$ via diagram (2) can be generalized to compact Lie groups in essentially two different ways, which we discuss next.

Decompositions via p–toral subgroups.

A p–toral group is a compact Lie group P whose component of the unit P_0 is a torus and whose group of components P/P_0 is a finite p–group. p–toral groups play the same role for compact Lie groups as finite p–groups do for finite groups. For example, every compact Lie group G has a p–toral Sylow subgroup $S_pG \subset G$. It has the same properties as a p–Sylow subgroup of a finite group; e.g. the group S_pG is maximal in the sense that every p–toral subgroup of G is subconjugate to S_pG. It is also characterized by the condition that the Euler characteristic of G/S_pG is coprime to p. Let $T_G \subset G$ be a maximal torus of G and let $N(T_G) \to W_G$ be the projection of the normalizer of T_G onto the Weyl group of G. Then, the counter image of a p–Sylow subgroup S_pW_G of W_G is a p–toral Sylow subgroup of G.

For any compact Lie group G, let $\mathcal{O}(G)$ denote the (topological) orbit category, whose objects are homogenous spaces G/H with $H \subset G$ being a closed subgroup and whose morphisms are given by G–equivariant maps. Let $\mathcal{O}_p(G) \subset \mathcal{O}(G)$ denote the full subcategory of all objects G/P, where $P \subset G$ is a p–toral subgroup. Let $\mathcal{I} : \mathcal{O}_p(G) \to \mathcal{T}op$ be the inclusion functor. Then, the Borel construction defines a (continuous) functor

$$EG \times_G \mathcal{I} : \mathcal{O}_p(G) \to \mathcal{T}op\ .$$

Notice that the $EG \times_G G/P \simeq BP$.

If G is a finite group, the category $\mathcal{O}_p(G)$ is finite (and so is $\mathcal{O}$). In this case, the map $\underset{\mathcal{O}_p(G)}{hocolim}\, EG \times_G \mathcal{I} \to BG$ is a mod–p equivalence, since all higher inverse limits in the associated spectral sequence of Theorem A.1 vanish [Mis] and since the inverse limit involved equals the mod–p cohomology of BG [C–E; XII, 10.1].

For compact Lie groups, the category $\mathcal{O}_p(G)$ is not finite and not even discrete in general. For a generalization of the above result, the question comes up, which of the p–toral subgroups cannot be got rid of in a decomposition of BG. This motivates the notion of p–stubborn subgroups. More concretely, a p–toral subgroup is called p–stubborn if the quotient $N(P)/P$ of the normalizer of P by P is finite and does not contain any nontrivial normal p–subgroup. Let $\mathcal{R}_p(G) \subset \mathcal{O}_p(G)$ denote the full subcategory of all objects G/P where P is p–stubborn. This turned out to be a finite category [J–M–O 1]. Restricting the above functor to this subcategory, Jackowski, McClure and Oliver proved the following decomposition theorem using techniques from the theory of transformation groups.

1.2 Theorem. *([J–M–O 1]) For any compact Lie group G, the map*

$$\underset{\mathcal{R}_p(G)}{hocolim}\, EG \times_G \mathcal{I} \to BG$$

is a p–local equivalence, i.e. induces an isomorphism in $H^*(-;\mathbb{Z}_{(p)})$ cohomology.

Decompositions via centralizers of elementary abelian subgroups.

For any compact Lie group G, let $A_p(G)$ denote the Quillen category [Qu 1] whose objects are nontrivial elementary abelian p–subgroups $V \subset G$ and whose morphisms are given by restrictions of conjugations by elements of G. Actually, Quillen also allowed the trivial group to be an object of $A_pP(G)$ but we exclude it. Let

$$\beta : A_p^{op}(G) \to \mathcal{T}op$$

be the functor given by the Borel construction $\beta(V) := EG \times_G G/C_G(V)$, where $C_G(V)$ denotes the centralizer of V in G and where G acts on $G/C_G(V)$ via left translation. Starting from the opposite category of $A_p(G)$ makes β into a covariant functor.

The projection $G/C(V) \to *$ to a point establishes a natural transformation from β to the constant functor with value BG and a map $\underset{A_p(G)}{hocolim}\, \beta \to BG$. These constructions were used by Jackowski and McClure to get a decomposition of BG into simpler pieces.

1.3 Theorem. *(*[J–M 2]*) Let G be compact Lie group. Then the map*

$$\underset{A_p(G)}{hocolim}\, \beta \to BG$$

is a mod–p equivalence.

The proof is based on the spectral sequence of Theorem A.1. Using transfers and Feshbach's double coset formula [Fe], one first proves that $H^*(BG;\mathbb{F}_p) \cong \underset{\overleftarrow{A_p(G)}}{\lim}\ H^*(\beta(-);\mathbb{F}_p)$ is an isomorphism. The proof of the vanishing of the higher derived functors of the inverse limit functor is also based on the existence of a transfer for the functor $H^*(\beta(-);\mathbb{F}_p)$. This functor turns out to be a Mackey functor in a sense closely related to the definition given in [Dr]. This extra structure allows the proof to be completed.

This geometric decomposition was generalized by Dwyer and Wilkerson [D–W 3]. They formulated Theorem 5.1 in purely algebraic terms using mod–p cohomology and also gave an algebraic proof of this theorem based on Lannes' T–functor.

Let $\mathcal{K}$ denote the category of unstable algebras over the Steenrod algebra. For any object $R \in \mathcal{K}$, Rector defined a category $A(R)$ [Re 3]. The objects are given by morphisms $\phi_V : R \to H^*(BV;\mathbb{F}_p)$ such that $H^*(BV;\mathbb{F}_p)$ is a finitely generated module over R and such that V is a nontrivial elementary abelian group. The morphisms are given by commutative triangles

$$\begin{array}{ccc} & R & \\ \swarrow & & \searrow \\ H^*(BV;\mathbb{F}_p) & \longrightarrow & H^*(BW;\mathbb{F}_p) \end{array} \quad .$$

Lannes' T–functor defines a functor

$$\tau : A(R) \to \mathcal{K}$$

given by $\tau(\phi_V) := T_V(R,\phi_V)$.

As a consequence of Lannes' theory (see Appendix B), of a theorem of Dwyer and Zabrodsky [D–Z] (see Theorem 2.1) and of a result of Quillen [Qu 1] (see 5.2), passing to mod–p cohomology establishes an isomorphism $A_p(G) \cong A(H^*(BG;\mathbb{F}_p)$ of categories and a natural equivalence $H^*(\beta(-);\mathbb{F}_p) \xrightarrow{\cong} \tau$. To reprove Theorem 5.1 in algebraic terms, Dwyer and Wilkerson also used the existence of a p–toral Sylow subgroup $P \subset G$, which, by analogy to the p–Sylow group of a finite group, has the properties that $H^*(BG;\mathbb{F}_p) \to H^*(BP;\mathbb{F}_p)$ is a monomorphism, that this map has a left inverse as $H^*(BG;\mathbb{F}_p)$–module homomorphism given by the transfer, and that P has a central subgroup. Translating group theory notions into mod–p cohomology we say that a morphism $\phi_V : R \to H^*(BV;\mathbb{F}_p)$ of $A(R)$ is central if $T_V(R,\phi_V) \cong R$ and that R has a nontrivial center if there exists a central morphism in $A(R)$. Now, Theorem 5.1 can be reformulated to

1.4 Theorem. *([D–W 3]) Let $i : R \to S$ be a morphism in $\mathcal{K}$ such that the following holds:*

(1) *R and S are noetherian algebras.*
(2) *There exists a left inverse $S \to R$ of i which is both a map of R–modules and a map over the Steenrod algebra.*
(3) *The algebra S has a nontrivial center.*

Then $\lim\limits_{\overleftarrow{A(R)}} \tau \cong R$ and $\lim\limits_{\overleftarrow{A(R)}}{}^i \tau$ vanishes for $i \geq 1$.

For any compact Lie group G, the mod–p cohomology $H^*(BG;\mathbb{F}_p)$ is noetherian [Ve]. Hence, the first condition is also satisfied in the case of $R = H^*(BG;\mathbb{F}_p)$.

For objects of $\mathcal{K}$ with nontrivial center, it turns out that the center (which is fixed under "conjugations") plays the role of an initial element in the category $A(R)$. This makes the proof in this case possible. For the general case, the higher inverse limits of τ defined on $A(R)$ and on $A(S)$ are compared. Using the exactness of the T–functor, it can be shown that the higher derived inverse limits taken over $A(R)$ are a direct summand of the ones over $A(S)$.

This algebraic proof of the geometric decomposition theorem allows a generalization to a much larger class of spaces than just compact Lie groups. One only needs that passing to mod–p cohomology controls a sufficient part of the homotopy theory of a given space (for explicit conditions see [D–W 3]). In particular, decompositions of this type exist for p–compact groups (see Theorem 5.14).

The important role which the Quillen and the Rector categories play in the homotopy theory of classifying spaces inspired Oliver to analyse inverse limits of general functors from the Rector category into the category of abelian groups [Ol 2]. He set up a spectral sequence converging against the higher limits and computed the E^2–term in terms of the endomorphism sets of single objects. Because every endomorphism of the Rector category is an isomorphism, the objects form a poset. This gives rise to a filtration of the functor such that the quotients are nontrivial only on one particular object. Then, the E^2-term is given by the higher limits of these atomic functors which turn out only to depend on the automorphism set of this object. In particular, Oliver showed that, for any noetherian algebra K over the Steenrod algebra and any functor $F : \mathcal{A}(K) \to \mathcal{A}b$, all higher limits vanish above a certain degree.

2. Maps between classifying spaces.

Sullivan [Su] (for $BU(n)$) and later Wilkerson [Wi 1] (in the general case) constructed self maps of classifying spaces of connected compact Lie groups, which are called unstable Adams operations. That is a self map $f : BG \to BG$ which, for a suitable $k \in \mathbb{N}$, induces multiplication by k^i in the rational cohomology group $H^{2i}(BG;\mathbb{Q})$. In this case we say that f has degree k. The name comes from the fact that f induces in complex K–theory a map which looks like an Adams operation of degree k. These examples destroyed the hope that, up to homotopy, all maps between classifying spaces are induced by homomorphisms.

These examples also motivated Adams, Mahmud and Hubbuck [A–M 1,2] [Ad] [Hub 1,2] to study carefully the effect maps between classifying space could have in rational cohomology. Methods and results, which are available today and which are consequences of the generalized Sullivan conjecture (Theorem C.1), allow a more precise analysis of such maps. Results of great importance are those of Dwyer and Zabrodsky [D–Z] and of the author [No 2].

In contrast to the above mentioned examples of Sullivan and Wilkerson, maps $BP \to BG$ are always induced by homomorphisms if P is a p–toral group and G a compact Lie group. To be more explicit, let $Rep(P,G) := Hom(P,G)/Inn(G)$ denote the set of representations $P \to G$, i.e. the set of all homomorphisms $P \to G$ modulo inner automorphisms of G.

2.1 Theorem. [D–Z] [Za 4] [No 2] *Let P be a p–toral group and G a compact Lie group. Then, passing to classifying spaces induces a bijection*

$$Rep(P,G) \to [BP,BG] \ .$$

Moreover, for any homomorphisms $\rho : P \to G$, there exists equivalences

$$BC_G(\rho(P))^\wedge_p \xrightarrow{\simeq} (map(BP,BG)_{B\rho})^\wedge_p \simeq map(BP,BG^\wedge_p)_{B\rho^\wedge_p} \ .$$

The map is given by the adjoint of $BC_G(\rho(P)) \times BP \simeq B(C_G(P)) \times P) \to BG$ induced by the homomorphism ρ. The second equivalence comes from the technical dainty that in this case passing to completion and taking mapping spaces commute (see [J–M–O 1, Theorem 3.2] and for a general statement [B–N]). In the first part of the statement we have to divide out conjugations by elements of G , because for any compact Lie group inner automorphisms induce self maps on the classifying space homotopic to the identity [Se]. For finite p–groups Theorem 2.1 was proved by Dwyer and Zabrodsky, and in the general case by the author. Zabrodsky also found a proof for tori.

For an outline of the proof let us assume that P is a finite p–group. The second part of the theorem is a consequence of the generalized Sullivan conjecture (Theorem C.1). Taking loops in the map $BC_G(\rho(p)) \to map(BP,BG^\wedge_p)_{B\rho}$ gives the fixed–point set G^P for the source and the homotopy fixed point set $G_p^{\wedge hP}$ for the target. Here, the group P acts on G via the homomorphism ρ and conjugation. For G^P this is obvious, and for $G_p^{\wedge hP}$ this follows from the observation that the loop space $\Omega map(BP,BG)_{B\rho}$ is equivalent to the space of sections of the pull back fibration of the free loop space fibration $\Lambda BG \to BG$ along the map $BP \to BG$ and the fact that the free loop space fibration is fiber homotopy equivalent to the fibration $EG \times_G G \to BG$ where G acts via conjugation on itself. The proof of the first part goes by an induction over the order of P. The starting point is given by the case $P = \mathbb{Z}/p$. In this case Lannes' theory is available and gives a way to calculate $[B\mathbb{Z}/p, BG]$ in terms of representations. In the induction step it only remains left to calculate the set of homotopy classes $[BP, BG]$. This is done using obstruction theory and by describing $\coprod_{Rep(P,G)} BC_G(\rho(P))$ and $map(BP, BG)$ as homotopy fixed point sets of $\mathbb{Z}/p$–actions on suitable spaces, which we can apply to the induction hypothesis. The step is based on the observation that any finite p–group P fits into a short exact sequence $1 \to P_0 \to P \to \mathbb{Z}/p \to 1$.

In the case of P being a p–toral group, one uses the mod–p approximation of P by its finite p–subgroups to achieve a generalization of the generalized Sullivan conjecture [No 1] and a proof of Theorem 2.1.

Theorem 2.1 also allows the following corollary.

2.2 Corollary. [No 2] *Let T be a torus and G a compact Lie group. Then, passing to rational cohomology induces an injection*

$$[BT, BG] \to Hom(H^*(BG;\mathbb{Q}), H^*(BT;\mathbb{Q})) \ .$$

And using this corollary we can reformulate a theorem of Adams and Mahmud as

2.3 Theorem. [A–M 1] *Let G and H be two connected compact Lie groups with maximal tori T_G and T_H. Then, for every map $BG \to BH$, there exists a homomorphism $\alpha : T_G \to T_H$ such that the diagram*

$$\begin{array}{ccc} BT_G & \xrightarrow{B\alpha} & BT_H \\ \downarrow & & \downarrow \\ BG & \xrightarrow{f} & BH \end{array}$$

commutes up to homotopy. Moreover, if $\beta : T_G \to T_H$ is another homomorphism with this property, then we have $\beta = w' \circ \alpha$ for some element $w' \in W_H$.

Adams and Mahmud actually proved that the diagram commutes in rational cohomology, and that the last identity also holds in rational cohomology. But by Corollary 2.2 this is an equivalent statement. This theorem also led to the notion of admissible homomorphisms. A homomorphism $\alpha : T_G \to T_H$ is called admissible if for each $w \in W_G$ there exists a $w' \in W_H$ such that $w' \circ \alpha = \alpha \circ w$ (notice that for every $w \in W_G$ the composition $\alpha \circ w$ satifies Theorem 2.3 if α does).

Actually, this is a stronger definition than the one of Adams and Mahmud. They were interested in maps which exist after localisation at a set of primes and compared them with linear maps between the universal covers of the maximal tori.

Based on Theorem 2.1 and Theorem 2.3, Jackowski, McClure and Oliver set up a program to attack the classification of homotopy classes of maps between classifying spaces [J–M–O 1]. This program splits into several steps, which we explain next. For a much more detailed survey of this program we refer the reader to [J–M–O 4]. For the following G and H denote connected compact Lie groups.

Step 1: admissible homomorphisms.

By Theorem 2.3 every map $BG \to BH$ gives rise to a W_H–conjugacy class of an admissible homomorphism. For an admissible homomorphism $\alpha : T_G \to T_H$, let

$[BG, BH]_\alpha$ denote the set of homotopy classes of maps $BG \to BH$ which all give rise to the W_H–conjugacy class of α. Then we have $[BG, BH] = \coprod_\alpha [BG, BH]_\alpha$ where we take the union over all admissible homomorphisms α. The question comes down to a classification of all admissible homomorphisms and a study of the sets $[BG, BH]_\alpha$.

Adams and Mahmud proved that a homomorphism $\alpha : T_G \to T_H$ is admissible if and only if $B\alpha^* : H^*(BT_H; \mathbb{Q}) \to H^*(BT_G; \mathbb{Q})$ maps the W_H–invariants $H^*(BT_H; \mathbb{Q})^{W_H} \cong H^*(BH; \mathbb{Q})$ into the W_G–invariants
$H^*(BT_G; \mathbb{Q})^{W_G} \cong H^*(BG; \mathbb{Q})$ [A–M 1], which gives a necessary and sufficient condition to check the first part. The following steps deal with the second part of the problem.

Step 2: Passing to completions.

If we want to use the mod–p decomposition of BG given by Theorem 1.3, we can only analyse maps $BG \to BH_p^\wedge$ into the p–adic completion. Sullivan's arithemitic square [Su] [B–K] gives a way to pass forward and back between global data on the one side and mod–p and rational data on the other side. Because BH is simply connected and because BH is rationally a product of Eilenberg–MacLane spaces of even degrees, these techniques allow a proof of

2.4 Proposition. [J–M–O 1, Theorem 3.1] *Let G and H be connected compact Lie groups. For each admissible homomorphism $\alpha : T_G \to T_H$, the map*

$$[BG, BH]_\alpha \to \prod_{p \mid |W_H|} [BG, BH_p^\wedge]_\alpha$$

is a bijection.

If p does not divide the order $|W_H|$ of the Weyl group W_H, then there always exists an extension of α to a map $BG \to BH_p^\wedge$, unique up to homotopy (see Theorem 2.6). This is the reason why we only have to take into account those primes which divide $|W_H|$.

So we are left with the problem of calculating the sets $[BG, BH_p^\wedge]_\alpha$.

Step 3 : $\mathcal{R}_p(G)$–invariant representations.

For this step we fix a prime p. Let $N_p(T_G) \subset N(T_G) \subset G$ be a p–toral Sylow subgroup of G; i.e. $N_p(T_G)/T_G \subset W_G$ is a p–Sylow subgroup of W_G.

For a map $f : BG \to BH$, Theorem 2.1 provides more information than just the existence of an admissible homomorphism. The restriction $f|_{BN_p(T_G)} \simeq B\rho$ is homotopic to a map induced by a homomorphism $\rho : N_p(T_G) \to H$ which is unique up to conjugation in H (and this also is true for any p–toral subgroup). In particular, for any pair $P_1, P_2 \subset G$ of p–toral subgroups and subconjugations $c_{g_i} : P_i \to N_p(T_G)$ and any subconjugation $c_g : P_1 \to P_2$, the compositions $\rho \circ c_{g_2} \circ c_g$ and $\rho \circ c_{g_1}$ are conjugate in H. That is to say that the homomorphism ρ establishes an element

$$\hat{\rho} := (\rho_P) \in \lim_{\substack{\longleftarrow \\ G/P \in \mathcal{R}_p(G)}} Rep(P, H) \ .$$

Homomorphisms $N_p(T_G) \to H$ with this property are called $\mathcal{R}_p(G)$–invariant representations of $N_p(T_G)$. Every map $BG \to BH_p^\wedge$ which comes from an integral map gives rise to such an $\mathcal{R}_p(G)$-invariant representation. Thus the problem is now: given an admissible map $\alpha : T_G \to T_H$, does there exist an extension to an $\mathcal{R}_p(G)$–invariant representation $\rho : N_p(T_G) \to H$? And if so, how many conjugacy classes are there?

There is a lack of general techniques for doing this. But in the case of H being a classical matrix group like $U(n)$, $SU(n)$, $O(n)$ or $Sp(n)$, character theory is sufficient to check if two homomorphisms are conjugate and therefore, if a homomorphisms $N_p(T_G) \to H$ is $\mathcal{R}_p(G)$–invariant. If G is connected, every element of G is subconjugate to T_G. Thus, if H is one of the above mentioned groups character theory also tells us that there is at most one $\mathcal{R}_p(G)$–invariant extension of a given admissible homomorphism.

Finally we have to pass from $\mathcal{R}_p(G)$–invariant representations to actual maps.

Step 4 : From $\mathcal{R}_p(G)$–invariant representations to actual maps.

For this step we fix a prime p and a $\mathcal{R}_p(G)$–invariant representation $\rho : N_p(T_G) \to H$ respectively an element $\hat{\rho} = (\rho_P)_{G/P} \in \varprojlim_{G/P \in \mathcal{R}_p(G)} Rep(P, G)$. Let $map(BG, BH^\wedge_p)_\rho$ denote the union of all components given by the counterimage of ρ under the obvious map $[BG, BH] \to \varprojlim_{G/P \in \mathcal{R}_p(G)} Rep(G)$. Using the decomposition of BG of Theorem 1.3, we get $map(BG, BH^\wedge_p)_\rho \simeq map(\operatorname{hocolim}_{\mathcal{R}_p(G)} EG \times_G \mathcal{I}, BH^\wedge_p)_\rho$, and applying Theorem A.2 establishes a spectral sequence calculating the homotopy of $map(BG, BH^\wedge_p)_\rho$. Let

$$\Pi^\rho_1 : \mathcal{R}_p(G) \to p-groups \qquad \text{and} \qquad \Pi^\rho_n : \mathcal{R}_p(G) \to \mathcal{A}b$$

denote the functors given by

$$\pi^\rho_n(G/P) := \pi_n(map(BP, BH^\wedge_p)_{B\rho_P}) \cong \pi_n(BC_H(\rho_P)^\wedge_p) \ .$$

(Note that for any p–toral subgroup $P \subset G$ the group of components of the centralizer $C_H(\rho_P)$ is a finite p–group if $\pi_0(G)$ is one. [J–M–O 1; Proposition A.4].) Now Theorem A.1 and Corollary A.2 take the form

2.5 Theorem. [J–M–O 1] *Let $\rho : N_p(T_G) \to H$ be a $\mathcal{R}_p(G)$-invariant representation. Then, there exists a spectral sequence*

$$E_2^{p,q} := \varprojlim_{G/P \in \mathcal{R}_p(G)}{}^p \ \Pi^\rho_q \Longrightarrow \pi_{q-p}(map(BG, BH^\wedge_p)_\rho$$

which strongly converges. In particular, the map $B\rho$ has an extension $f : BG \to BH^\wedge_p$ if $\varprojlim_{G/P \in \mathcal{R}_p(G)}{}^{n+1} \ \Pi^\rho_n$ vanishes for all $n \geq 1$, and there exists at most one extension if $\varprojlim_{G/P \in \mathcal{R}_p(G)}{}^{n} \ \Pi^\rho_n$ vanishes for all $n \geq 1$.

The strong convergence follows from the fact that there exists an N, depending only on G, such that the n-th higher limit of any functor defined on $\mathcal{R}_p(G)$ vanishes for $n \geq N$ [J–M–O 1].

Now one has to face the analysis of the spectral sequences, i.e. in the first place the calculation of the higher limits. Although this looks like a very difficult and hard question, Jackowski, McClure and Oliver developed techniques to attack this problem successfully in many interesting cases (see [J–M–O 1,2,3]).

We demonstrate the power of this machinery in two cases.

2.6 Theorem. [J–M–O 4] *If G is connected and if $(p,|W_G|)=1$, then any admissible homomorphism $\alpha : T_G \to T_H$ has an extension $f : BG \to BH^\wedge_p$, unique up to homotopy.*

Proof. Actually this was already proved by Adams and Mahmud [A–M] with different methods. For a proof, the present theory can be used as follows. Because $(p,|W_G|)=1$, we have $T_G = N_p(T_G)$, and the category $\mathcal{R}_p(G)$ consists only of the object G/T_G. The set of endomorphisms is given by W_G. Hence, for any admissible homomorphism $\alpha : T_G \to T_H$, there exists a unique $\mathcal{R}_p(G)$–invariant representation $\rho = \alpha : N_p(T_G) = T_G \to H$. The higher limits are isomorphic to the cohomology groups $H^p(W_G; \pi_q(BT_{G}{}^\wedge_p))$ and vanish for $p \geq 1$. The associated spectral sequence of Theorem A.1 collapses, which finishes the proof. □

The other case, which is much more difficult and much deeper concerns integral self maps. Using their machinery, Jackowski, McClure and Oliver proved the following beautiful classification theorem for self maps of classifying spaces of simple connected compact Lie groups.

2.7 Theorem. [J–M–O 1] *Let G be a simple connected compact Lie group. Then there exists a bijection*

$$\Phi : [BG, BG] \to \{0\} \coprod (Out(G) \times \{k \geq 1 : (k,|W_G|)=1\} \ .$$

For two self maps $f, g : BG \to BG$, the following conditions are equivalent:

1. *(1) f and g are homotopic.*
2. *(2) The restrictions $f|_{BT_G}$ and $g|_{BT_G}$ are homotopic.*
3. *(3) The induced maps $H^*(f;\mathbb{Q})$ and $H^*(g;\mathbb{Q})$ in rational cohomology are equal.*

Moreover, for each map $f : BG \to BG$ and each prime there exist equivalences

$$BZ(G)^\wedge_p \simeq map(BG, BG)_{f}{}^\wedge_p \simeq map(BG, BG^\wedge_p)_{f^\wedge_p} \ .$$

Outline of the proof. The first step consists of a characterization of all admissible homomorphisms $T_G \to T_G$, which is due to Hubbuck [Hub 2] and Ishiguro [Is]. Hubbuck showed that, for G simple, every self map of BG looks in rational cohomology like a composition of a map induced by an outer automorphism and an unstable Adams operation of degree ≥ 0. Ishiguro proved that, for a connected compact Lie group G, every unstable Adams operation $BG \to BG$ has a degree coprime to the order of the Weyl group. Putting these facts together and passing to rational cohomology establishes the map Φ of the statement.

The existence of unstable Adams operations of any degree coprime to $|W_G|$ was shown by Sullivan [Su], Wilkerson [Wi 1] and Friedlander [Fr]. Hence, the map Φ is surjective, and it only remains to show that rational cohomology detects homotopy classes of self maps, or, what is sufficient, that rational cohomology detects homotopy classes of unstable Adams operations. One has to distinguish between the degree 0 and degrees ≥ 1. We only consider the latter case; the first one can be treated similiarly.

The associated admissible homomorphism of an unstable Adams operation of degree $k \geq 1$ is given by $\alpha_k : T_G \to T_G : t \mapsto t^k$. In the next step one has to show that up to conjugation there exists at most one $\mathcal{R}_p(G)$–invariant representation

$\rho_k : N_p(T_G) \to G$. For most of the classical matrix groups this follows from character theory and in general this is done in [J–M–O 1; Proposition 3.5].

After passing to completions as described in Step 2, one finally has to calculate the higher limits

$$\lim_{\substack{\longleftarrow \\ G/P\in\mathcal{R}_p(G)}}{}^p \ \pi_q(map(BP, BG^\wedge_p)_{B(\rho_k|_P)}) \cong \lim_{\substack{\longleftarrow \\ G/P\in\mathcal{R}_p(G)}}{}^p \ \pi_q(BC_G(\rho_k(P))^\wedge_p) \ .$$

The equivalence is a consequence of Theorem 2.1. Using the fact that, for p–stubborn subgroups $P \subset G$, the centralizer $C_G(\rho_k(P))$ is equal to the center $Z(P)$ of P [J–M–O 1; Lemma 1.5] and using general techniques developed for the calculation of higher limits of functors on $\mathcal{R}_p(G)$, Jackowski, McClure and Oliver were able to prove the vanishing of all higher limits under consideration and to show that $\lim\limits_{\substack{\longleftarrow \\ G/P\in\mathcal{R}_p(G)}} \pi_q(BC_G(\rho_k(P))^\wedge_p) \cong \pi_q(BZ(G)^\wedge_p)$. Hence, the associated spectral sequence of Theorem A.1 collapses. This proves the statement. (For more details see [J–M–O 1,4].) □

!! For any connected compact Lie group G, Jackowski, McClure and Oliver applied the same method to self maps $BG \to BG$ which induce an isomorphism in rational cohomology. There also exists a complete characterisation of all admissible maps in terms of Dynkin diagram symmetries or outer automorphisms and unstable Adams operations [J–M–O 3] [Mø1] [No 4]. The analogous statement as in Theorem 2.7 is true [J–M–O 3]. In particular, homotopy classes of rational self equivalences are detected by rational cohomology as well as by the restriction to the maximal torus.

Based on Theorem 2.7, different proofs for the same results about rational self equivalences of classifying spaces of connected compact Lie groups are given by Møller [Mø2] and by the author [No 4].

For general self maps, the classification problem is much harder, because that involves the classification of all maps between classifying spaces of connected compact Lie groups. In their work, Jackowski, McClure and Oliver constructed examples contradicting all reasonable conjectures one could make; e.g. they found a pair of non homotopic maps $f, g : B(SO(3) \times SO(3)) \to BSO(25)$, whose restrictions to $BN(T_G)$ are homotopic. Hence, homotopy classes of maps between classifying spaces cannot be detected by any cohomology theory in general.

What is known beyond this point ? One can look at self maps $BG^\wedge_p \to BG^\wedge_p$ of completed classifying spaces. For connected compact Lie groups similar results are obtained. The "completion" of Corollary 2.2 and of the Adams–Mahmud theorem (Theorem 2.3) are proved by Adams and Woijtkowiak [A–Wo] and by Smith and the author [N–S 1], the completion of Theorem 2.6 by Woijtkowiak [Wo 3] and the completion of Theorem 2.7 is due to Jackowski, McClure and Oliver [J–M–O 3].

Møller studied self maps of classifying spaces of nonconnected compact Lie groups [Mø3]. Every compact Lie group G gives rise to a fibration $Fib(G) : BG_0 \to BG \to B\pi := B\pi_0(G)$. Every self map $f : BG \to BG$ establishes a self map $B\rho : B\pi \to B\pi$ which is induced by a homomorphism. A rational self equivalence of BG is defined to be a fiber self map $(f, B\rho_f)$, such that $f|_{BG_0}; BG_0 \to BG_0$ induces an isomorphism in rational cohomology. Let $\epsilon_{\mathbb{Q}}(BG)$ denote the monoid of all vertical homotopy classes of rational self equivalences of BG. By definition there is a monoid map

$$\epsilon_{\mathbb{Q}}(BG) \to \epsilon_{\mathbb{Q}}(BG_0) \times End(\pi)$$

where $End(-)$ denotes the set of endomorphisms of a group, and where $\epsilon_{\mathbb{Q}}(BG_0)$ is actually the monoid of ordinary homotopy classes of rational self equivalences. The second coordinate of the map takes image among homomorphisms because a vertical homotopy induces a pointed homotopy on the base.

Let $g := f|_{BG_0}$ be the restriction on the fiber. The pull back via $B\rho$ establishes an induced fibration $B\rho^* Fib(G)$. Møller also constructed a fibration $g_* Fib(G)$ by imitating the push out for groups. Let $\epsilon_{\mathbb{Q},G}(BG_0, B\pi) \subset \epsilon_{\mathbb{Q}}(BG_0) \times End(\pi)$ be the subset of all pairs (g, ρ) such that $B\rho^* Fib(G)$ and $g_* Fib(G)$ are fiber homotopy equivalent fibrations. Møller proved the following classification result for rational self equivalences of nonconnected compact Lie groups.

2.8 Theorem. [Mø3] *For every compact Lie group G , there exists a short exact sequence of monoids*

$$1 \to H^1(\pi_0(G); Z(G)) \to \epsilon_{\mathbb{Q}}(BG) \to \epsilon_{\mathbb{Q},G}(BG_0, B\pi_0(G)) \ .$$

The action of $\pi_0(G)$ on the center $Z(G)$ is induced by conjugation. Using the analogue of Theorem 2.7 for connected compact Lie groups, this gives a complete classification of homotopy classes of rational self equivalences of classifying spaces of compact Lie groups.

More recently, Jackowski and Oliver used the method described to analyse maps $BG \to BU(n)$. In fact, they stabilized such maps in the same sense as is done for honest representations in $U(n)$. They defined

$$\mathbb{K}(BG) := Gr(\coprod_{n \geq 0} [BG, BU(n)])$$

as the Grothendieck group of the monoid $\coprod_n [BG, BU(n)]$. The monoid structure is inherited from the Whitney sum of vector bundles. There is also a multiplication which comes from the tensor product of vector bundles. Of course, this definition makes perfect sense for topological spaces, and, if X is a finite CW–complex, then we have $\mathbb{K}(X) \cong K(X) = [X, \mathbb{Z} \times BU]$.

For every prime, restriction to a p–toral subgroup P establishes a map

$$R : \mathbb{K}(BG) \to \prod_{\substack{P \subset G \\ \text{all primes}}} \mathbb{K}(BP) \ .$$

Using Theorem 2.1 , for p–toral groups, we can identify $\mathbb{K}(BP)$ with the representation ring $R(P)$. Let $\mathcal{O}_{\mathcal{P}}(G) \subset \mathcal{O}(G)$ denote the full subcategory of the objects G/P, where P is a p–toral group for some prime. So, what is the image and the kernel of R ?

Passing from homomorphisms to maps between classifying spaces and from $BU(n)$ to BU gives rise to a sequence of maps

$$R(G) \to \mathbb{K}(BG) \to K(BG).$$

from classical representation theory to $\mathbb{K}(BG)$ and to complex K–theory.

Let $I_R(G)$ denote the augmentation ideal of $R(G)$. Then, Atiyah [At 1] (for finite groups) and Atiyah and Segal [A–S] (for general compact Lie groups) showed that $I_R(G)$–adic completion induces an isomorphism $R(G)^\wedge_{I_R(G)} \xrightarrow{\cong} K(BG)$. How does the group $\mathbb{K}(BG)$ fit into this picture? Functorial properties establish a commutative diagram

$$(3)\qquad \begin{array}{ccc} R(G) & \xrightarrow{\lambda_G} & R(G)^\wedge_{I_R(G)} \\ {\scriptstyle \overline{\alpha}_G}\downarrow \cong & \searrow^{\alpha_G} & \downarrow {\scriptstyle \alpha_G{}^\wedge_I} \\ \mathbb{K}(BG) & \xrightarrow{\beta_G} & K(BG) \end{array}$$

The following statement, due to Jackowski and Oliver, answers the above questions.

2.9 Theorem. [J–O] *For every compact Lie group G, the map R induces an isomorphism $\mathbb{K}(BG) \xrightarrow{\cong} \varprojlim_{\mathcal{O}_\mathcal{P}(G)} R(P)$. Furthermore, in diagram (3), we have $ker(\lambda_G) = ker(\overline{\alpha}_G)$ and β_G is a monomorphism.*

The kernel of λ_G is known. It is given by all representations of G whose restriction to p–th power elments of G is trivial. For example, for connected compact Lie groups, λ_G as well as $\overline{\alpha}_G$ are monomorphisms. Jackowski and Oliver also computed the image of β_G and identified it with the formally finite elements of $K(BG)$ (see [Ad]). These are elements of $K(BG)$ which are mapped on 0 by λ–operations of large degree.

As usual, when proving a statement about homotopy classes of maps, the full mapping space has to be considered. There is a parallel construction leading to a Grothendieck group $\mathcal{K}(BG) := Gr(\coprod_n map(BG, BU(n)))$. The disjoint union $\coprod_n map(BG, BU(n)$ gets a monoid structure from the map $BU(n) \times BU(m) \to BU(n+m)$ which is induced from the Whitney sum of vector bundles. Then Jackowski and Oliver used a refinement of the above general approach to calculate the homotopy groups of $\mathcal{K}(BG)$. These calculations give a proof of Theorem 2.9.

Jackowski and Oliver also looked at real vector bundles over classifying spaces of compact Lie groups and got similar results as in Theorem 2.9 [J–O].

3. The Steenrod problem : Realizations of polynamial algebras.

Steenrod posed the question, which polynomial algebras over $\mathbb{F}_p$ appear as the mod–p cohomology of a topological space [St]? Examples are provided by classifying spaces of connected compact Lie groups. For a connected compact Lie group G the mod–p cohomology $H^*(BG;\mathbb{F}_p)$ is polynomial for almost all primes (in particular for primes coprime to the order of the Weyl group and in several cases for all primes). If this is the case, a result of Borel [Bo 1] tells us that, at least for odd primes, the inclusion of the maximal torus $T_G \to G$ induces an isomorphism $H^*(BG;\mathbb{F}_p) \cong H^*(BT_G;\mathbb{F}_p)^{W_G}$. For primes not dividing the order of W_G, a straightforward calculation of a Serre spectral sequence shows that the map $BN(T_G) \to BG$ is a mod–p equivalence. This observation led Clark and Ewing to a construction of several exotic examples of spaces with polynomial cohomology [Cl–Ew]. They considered finite pseudo reflection groups $W \to Gl(n, \mathbb{Z}_p^\wedge)$, whose order is coprime to p. That is the map is a monomorphism and W is generated by pseudo reflections. And a pseudo reflection is a linear map of finite order, which fixes a hyperplane of codimension 1. Every such group induces an action on the

Eilenberg–MacLane space $K := K(\mathbb{Z}_p^{\wedge n}, 2)$. Then, the mod–p cohomology of the Borel construction $EW \times_W K$ is given by the invariants and is polynomial by a theorem of Chevalley [Ch]. Here, exotic means that these spaces are not equivalent to the completion of a classifying space of a compact Lie group.

Later Adams and Wilkerson gave criteria which ensure that a polynomial algebra on n-generators over the Steenrod algebra is isomorphic to the invariants of a pseudo reflection group $W \to Gl(n, \mathbb{F}_p)$ acting on the polynomial part P_V of $H^*((B\mathbb{Z}/p)^n; \mathbb{F}_p)$ [A–W]. For odd primes, Dwyer, Miller and Wilkerson showed that, for a polynomial algebra $P \cong H^*(X; \mathbb{F}_p)$ given by the mod–p cohomology of a space, these conditions are always satisfied and that the associated pseudo reflection group $W \to Gl(n; \mathbb{F}_p)$ always lifts to $Gl(n; \mathbb{Z}_p^\wedge)$ [D–M–W 2].

3.1 Theorem. [A–W] [D–M–W 2] *Let p be an odd prime. Let P b a polynomial algebra over the Steenrod algebra. If $P \cong H^*(X; \mathbb{F}_p)$, then there exists a pseudo reflection group $W \to Gl(n, ; \mathbb{Z}_p^\wedge)$ such that $P \cong (P_V)^W$.*

Based on this result, Dwyer and Wilkerson proved the following realization and uniqueness theorem for plynomial allgebras.

3.2 Theorem. [D–M–W 2] *Let P be a polynomial algebra over the Steenrod algebra generated by elements of degree coprime to p. Then there exists a p–complete space X, unique up to homotopy, with $H^*(X; \mathbb{F}_p) \cong P$.*

All these algebras are realized by the examples of Clark and Ewing. For a proof of the uniqueness see the proof of Theorem 4.2.

Theorem 3.1 also shows that, for odd primes, a solution of Steenrod's problem asks for a classification of pseudo reflection groups over the p–adic integers. Clark and Ewing gave a complete list of all p–adic rational irreducible reflection groups $W \to Gl(U)$ where U is a vector space over the p-adic rationals. So, slightly changing the problem, one might ask for a realization of these irreducible p–adic rational pseudo reflection groups. Notice that the classifying space of every simple connected compact Lie group realizes one of the irreducible pseudo reflection groups, but not every of these spaces has polynomial mod–p cohomology.

Besides the Clark–Ewing spaces, computations of Quillen on the mod-p group cohomology of general linear groups over finite fields of characteristic coprime to p [Qu 2] and adhoc constructions of Zabrodsky [Za 1] gave further spaces whose mod–p cohomology is polynomial. In these cases the order of the associated pseudo reflection group is not coprime to p, which makes constructions much more difficult. The examples of Quillen and Zabrodsky as well as the examples we discuss next realize irreducible pseudo reflection groups.

More recently, Aguadé [Ag] and Dwyer and Wilkerson [D–W 4] approached the Steenrod question using ideas from the decomposition theorems for classifying spaces. Aguadé looked at diagrams similar to the decomposition diagram of $BSO(3)$ (see Section 1, diagram (2)). For a pair of groups $H \subset G$ he considered the category $\mathcal{C}(G, H)$ with two objects 0 and 1 and morphism sets given by $End(0) = G$, $End(1) = \{1\}$, and $Hom(0, 1) = G/H$ and $Hom(1, 0) = \emptyset$, constructed a "nice" functor $\mathcal{C}(G, H) \to \mathcal{T}op$ into the category of topological spaces and took the homotopy limit of F. The Bousfield–Kan spectral sequence of Theorem A.1 gives a tool to compute the mod–p cohomology of the homotopy colimit. Hopefully all higher limits involved vanish.

3.3 Theorem. [Ag] *Let $H \subset G$ be a pair of finite groups. If, for any $\mathbb{F}_p[G]$–module M, restriction induces an isomorphism $H^*(G,M) \cong H^*(H,M)$, then there exists a functor $F : \mathcal{C}(G,H) \to Top$ such that $H^*(F(0);\mathbb{F}_p) =: P$ is a polynomial algebra generated by elements of degree 2, such that $H^*(F(1);\mathbb{F}_p) \cong P^H$ and such that $H^*(\underset{\mathcal{C}(G,H)}{hocolim}\, F;\mathbb{F}_p) \cong P^G$.*

Aguadé applied his result to several cases of the Clark–Ewing list, covering the examples of Zabrodsky and producing some new spaces with polynonial mod–p cohomology. He also reconstructed the classifying spaces of the exeptional Lie groups E_6, E_7, E_8 at those primes which do not appear as torsion primes in the integral homology of the particular group.

Dwyer and Wilkerson chose an approach based on the algebraic decomposition via centralizers of elementary abelian subgroups (Theorem 1.3 and Theorem 1.4). They found a space whose mod–2 cohomolgy is given by the Dickson invariants in dimension 4.

3.4 Theorem. [D–W 4] *There exists a space X with*

$$H^*(X;\mathbb{F}_2) \cong H^*((B\mathbb{Z}/2)^4;\mathbb{F}_2)^{Gl(4,\mathbb{F}_2)} =: D(4) \ .$$

Outline of the proof. The idea of the proof comes from the fact, that there exists an algebraic decomposition of $D(4)$ over the Rector category of $D(4)$ (Theorem 1.4) and that the topological realization should give a topological decomposition of such a space. For each object $\phi : D(4) \to H^*(BV;\mathbb{F}_2)$ of the Rector category $A_2(D(4))$, a calculation of the pieces $T_V(D(4);\phi)$ of the algebraic decomposition shows that these algebras are given by the mod–2 cohomology of $BC_{Spin(7)}(V)$ for a suitable inclusion $V \subset Spin(7)$. Here, V is an elementary abelian 2–group of dimension ≤ 4. In the next step Dwyer and Wilkerson constructed a functor $F : A_2(D(4)) \to Ho\mathcal{T}op$ into the homotopy category of topological spaces, which realizes the algebraic data. This is the hard part of the matter. Beside the solution in this special case, Dwyer and Wilkerson approached such questions in a more general context, including algebraic decomposition of spaces whose mod-p cohomolgy satisfies the assumptions of Theorem 1.4 [D–W 8].

Because homotopy colimits do not exist in the homotopy category, one has finally to find a lift $\tilde{F} : A_2(D(4)) \to \mathcal{T}op$ of F. For such a need, Dwyer and Kan had developped an obstruction theory [D–K 1,2]. In the case under consideration the obstruction groups are given by some higher limits of a functor on $A_2(D(4))$ with the homotopy groups of $map(\widetilde{F}(\phi),\widetilde{F}(\phi))_{id}$. as values. The functor F takes image among the classifying spaces of certain subgroups of $Spin(7)$ and, passing everywhere to completions, the mapping spaces can be identified with centers (see Theorem 2.7 and extensions). This allows a proof of the vanishing of the obstruction groups and of the existence of $\widetilde{F}$. The mod–2 cohomology of the homotopy colimit $\underset{A_2(D(4))}{hocolim}\,\widetilde{F}$ can be computed using the spectral sequence of Theorem A.1. Theorem 1.4 proves the vanishing of all higher limits involved. This finishes the proof. □

An Eilenberg–Moore spectral sequence argument shows that, for every space X with polynomial mod–p cohomology, the cohomology $H^*(\Omega X;\mathbb{F}_p)$ of the loop space is finite. That is that X is the classifying space of a connected p–compact group. In

Section 5, we show that every connected p–compact group comes with an associated pseudo reflection group $W \to Gl(n;\mathbb{Q}_p^\wedge$. This group W plays the same role as Weyl groups do for connected compact Lie groups. In this sense the example of Dwyer and Wilkerson gives a realization of another irreducible pseudo reflection group at the prime 2, although the group $Gl(4,\mathbb{F}_2)$ does not lift to $\mathbb{Z}frm{-}e$. ($Gl(4,\mathbb{F}_2)$ is not the Weyl group of the Dwyer–Wilkerson example.)

The Eilenberg–Moore spectral sequence argument also shows that the solution of Steenrod's problem is closely related to the classification of all connected p–compact groups, which we will discuss in Section 5.

4. Homotopy uniqueness of classifying spaces.

Connected compact Lie groups are very rigid objects. A few combinatorial data are sufficient to distinguish between two connected compact Lie groups; e.g. Dynkin diagrams classify the local isomorphism types of semi simple connected compact Lie groups, or the isomorphism types of simply connected compact Lie groups.

Surprisingly, classifying spaces of connected compact Lie groups also seem to be very rigid objects. The algebra $H^*(BG;\mathbb{F}_p)$ considered as an algebra over the Steenrod algebra determines the homotopy type of the p–adic completion $BG_p^\wedge$ in a large number of cases. This is what we mean by the homotopy uniqueness of the classifying spaces of connected compact Lie groups. The first results of this type were proved by Dwyer, Miller and Wilkerson [D–M–W 1,2]. We say that two spaces X and Y have the same mod–p type if $H^*(X;\mathbb{F}_p) \cong H^*(Y;\mathbb{F}_p)$ as algebras over the Steenrod algebra.

4.1 Theorem. *(*[D–M–W 1]*) Let $G = SU(2)$ or $G = SO(3)$ and let X be a p–complete space. Then the spaces X and BG have the same mod–p type if and only if they are homotopy equivalent.*

4.2 Theorem. *(*[D–M–W 2]*) Let G be a connected compact Lie group and let X be a p–complete space. Asume that $(p,|W_G|) = 1$, Then the two spaces X and BG have the same mod–p type if and only if they are homotopy equivalent.*

For $U(2)$, McClure and Smith proved the analogous result of Theorem 4.1 [McC–S]. The second theorem is a special case of Theorem 3.2 and also covers Theorem 4.1 for odd primes. To give the reader an idea of the techniques used for the proof we outline the proof of the second theorem. The main idea is to combine Lannes' theory (see Appendix B) and the Dwyer–Zabrodsky theorem (Theorem 2.1) into a powerful tool.

Proof of Theorem 4.2. By assumption, p is an odd prime not dividing the order of W_G. By [Bo 1], this implies that $H^*(BG;\mathbb{F}_p) \cong H^*(BT_G;\mathbb{F}_p)^{W_G}$. We fix such an isomorphism and try to realize it by a topological map. As a first step we construct a map $f : BT_G \to X$ which looks in mod–p cohomology like the map $Bi : BT_G \to BG$. By Theorem B.1, the composition $H^*(X;\mathbb{F}_p) \cong H^*(BG;\mathbb{F}_p) \to H^*(BV;\mathbb{F}_p)$ has a topological realization $f_V : BV \to X$ where $V \subset T_G$ is a maximal elementary abelian subgroup. Because p is odd, we have $C_G(V) = T_G$ for the centralizer of V in G [D–M–W 2]. The application of the T–functor and Theorem 2.1 establish a diagram

$$\begin{array}{ccccc} T_V(H^*(X;\mathbb{F}_p),f^*) & \longrightarrow & H^*(map(BV,X)_f;\mathbb{F}_p) & & \\ \cong\downarrow & & & & \\ T_V(H^*(BG;\mathbb{F}_p),Bi^*) & \longrightarrow & H^*(map(BV,BG_p^\wedge)_{f_V};\mathbb{F}_p) & \xrightarrow{\cong} & H^*(BT_G;\mathbb{F}_p)\ . \end{array}$$

Because $BT_{G_p}^\wedge$ is simply connected, the lower left arrow is an isomorphism (Theorem B.2) and so is the upper arrow, since $T_V(H^*(X;\mathbb{F}_p),f^*)$ vanishes in degree 1 (Theorem B.2). The mod-p cohomology determines the homotopy type of $BT_{G_p}^\wedge$. Hence, the mapping space $map(BV,X)_f$ and $BT_{G_p}^\wedge$ are equivalent. Again by Theorem B.2, the action of W_G on V fixes the component of f, for it does it cohomologically, and establishes a second action of W_G on $BT_{G_p}^\wedge$. Mod–p, both actions are equivalent, which follows from the above sequence of isomorphisms. For an appropriate basepoint of BV, the evaluation $map(BV,X)_f \to X$ induces the desired map $f : BT_{G_p}^\wedge \to X$, which is W_G–equivariant as well as Bi, where W_G acts trivially on BG and on X.

Because p is coprime to $|W_G|$, both actions of W_G on $BT_{G_p}^\wedge$ are equivalent over the p–adic integers. Passing to the Borel construction yields a map $EW_G \times_{W_G} BT_G \to X$. Again, because $p, |W_G|) = 1$ we have a sequence of maps $BG_p^\wedge \leftarrow BN(T_G)_p^\wedge \simeq (EW_G \times_{W_G} BT_G)_p^\wedge \to X$. A straight forward calculation of the mod–p cohomology shows that both arrows are homotopy equivalences, which finishes the proof. □

For $p = 2$ and $G = SO(3)$, Dwyer, Miller and Wilkerson used the pushout diagram of $BSO(3)$ described in Section 1 (diagram (1)). Given a space Y with the same mod–2 cohomology, they constructed maps from all pieces of this diagram into Y and showed that the associated diagram

$$\begin{array}{ccc} BD(8) & \longrightarrow & BO(2) \\ \downarrow & & \downarrow \\ B\Sigma_4 & \longrightarrow & Y \end{array}$$

commutes up to homotopy. This is the hard part of the proof and again based on combining Lannes' theory with the Dwyer–Zabrodsky theorem (notice that $O(2) = C_{SO(3)}(\mathbb{Z}/2)$). The above diagram establishes a homotopy equivalence $BSO(3)frm{-}e \to Y$. For $G = SU(2)$, the homotopy uniqueness is proved with the help of the sequence of fibrations $B\mathbb{Z}/2 \to BSU(2) \to BSO(3) \to B^2\mathbb{Z}/2$.

In general, mod–p cohomology is not sufficient to characterize the homotopy type of BG for connected compact Lie groups G; e.g. the spaces $B(SU(p^2) \times S^1)$ and $B(SU(p^2) \times_{\mathbb{Z}/p} S^1)$ have isomorphic mod–p cohomology, but are not homotopy equivalent [No 5; 9.6]. For non simply connected compact Lie groups one needs a little extra information. Let X be a p–complete space with the same mod–p type as BG. As shown above there exists a map $BT_{G_p}^\wedge \to X$ and another W_G–action on $BT_{G_p}^\wedge$ making the map equivariant (for $p = 2$ one needs an extra assumption (see [No 5]). We say that X has the p–adic type of BG if it has the same mod–p type and if both actions of W_G on $BT_{G_p}^\wedge$ are p–adically equivalent. Actually, this is a rough version of the technical definition given in [No 5] but hits the heart of the matter. We say that BG is p–torsion free if $H^*(BG;\mathbb{Z})$ has no p-torsion. In [No 5], the following homotopy uniqueness result is proved.

4.3 Theorem. *(*[No 5]*) Let p be an odd prime. Let G be a connected compact Lie group such that BG is p–torsion free. Let X be a p–complete space.*

1. *If X has the mod–p type of BG, then there exists a connected compact Lie group H such that X and BH have the same p–adic type.*
2. *The space X has the p–adic type of BG, if and only if X and $BG^\wedge_p$ are homotopy equivalent.*
3. *If G is simply connected, if G is a product of unitary groups, or if $(p, |W_G|) = 1$, then X has the mod–p type of BG if and only if X and $BG^\wedge_p$ are homotopy equivalent.*

For $p = 2$ similar results are true for quotients of products of unitary and special unitary groups, but one has to exclude $SU(2) = Sp(1)$ as factor. For odd primes, this covers all classical matrix groups and among the exeptional Lie groups only a few cases are missed (for a complete list see [No 5]).

The proof of Theorem 4.3 is heavily based on the work of Jackowski, McClure and Oliver, their decomposition of BG via p–stubborn subgroups and their analysis of self maps of BG. The idea is to construct the identity $id : BG^\wedge_p \to BG^\wedge_p$ purely by algebraic means. The assumption that BG is p–torsionfree is essential for the proof. In particular, it implies that $H^*(BG; \mathbb{F}_p) \cong H^*(BT_G; \mathbb{F}_p)^{W_G}$ [Bo 1] which makes calculation with the Lannes T–functor easier. Furthermore, Oliver's computation of the p–stubborn subgroups of the classical matrix groups [Ol 1] allows a sufficient understanding of the category $\mathcal{R}_p(G)$ in these cases. The proof also uses the classification of connected compact Lie groups. For simple simply connected Lie groups, the proof is done by a case by case checking and differs only in details from the one for $U(n)$.

To demonstrate the ideas we consider the case of $G = U(n)$ and an odd prime. We fix an isomorphism $H^*(X; \mathbb{F}_p) \cong H^*(BU(n); \mathbb{F}_p)$. As in the proof of Theorem 4.2, we construct a "maximal torus" $f_T : BT_{U(n)}{}^\wedge_p \to X$ and another action of $W_{U(n)} = \Sigma_n$ on $BT_{U(n)}{}^\wedge_p$. By construction, the two representations $\rho_{U(n)}, \rho_X : W_{U(n)} \to Gl(n; \mathbb{Z}^\wedge_p)$, associated to the two actions of $W_{U(n)}$ on $BT_{U(n)}{}^\wedge_p$, are equivalent mod–p. Because the standard permutation representation over $\mathbb{F}_p$ has only one p–adic lift [No 5: Proposition 11.1], the two representations are p–adically conjugate, and we assume they are equal. That is to say that $id : BT_{U(n)}{}^\wedge_p \to BT_{U(n)}{}^\wedge_p$ is an "admissible" map. (For a general connected compact Lie group there may exist several, but finitely many p–adic liftings, and each of them is associated with a possibly different connected compact Lie group.)

The classifying space $BN(T_{U(n)})$ of the normalizer of $T_{U(n)}$ can be thought of as the homotopy colimit of the diagram $\mathcal{D}$ given by the action of $W_{U(n)}$ on $EN(T_{U(n)})/T_{U(n)} \simeq BT_{U(n)}$. Then, by Corollary A.2, the obstruction for extending f_T to a map $f_N : BN(T_{U(n)}) \to X$ lie in the groups

$$\begin{aligned} H^{*+1}(W_{U(n)}; \pi_*(map(BT_{U(n)}, X)_{f_T}) &\cong H^{*+1}(W_{U(n)}, \pi_*(BT_{U(n)}{}^\wedge_p) \\ &\cong H^3(\Sigma_n; (\mathbb{Z}^\wedge_p)^n) \end{aligned}$$

which vanish for odd primes. In this case, the higher limits are given by group cohomology. The first isomorphism follows from a lemma we will mention in a moment.

Thus, the extension $f_N : BN(T_{U(n)}) \to X$ exists and for every object $U(n)/P \in \mathcal{R}_p(U(n))$ we define a map $f_P := f_N|_P : BP \to X$. We have to show that this gives rise to an $\mathcal{R}_p(U(n))$–invariant representation. First one shows that the triangle

$$\begin{array}{ccc} & H^*(BN(T_{U(n)});\mathbb{F}_p) & \\ \swarrow & & \searrow \\ H^*(X;\mathbb{F}_p) & \xrightarrow{\cong} & H^*(BU(n);\mathbb{F}_p) \end{array}$$

commutes. This is based upon proving that the map $H^*(BU(n);\mathbb{F}_p) \to H^*(BT_{U(n)};\mathbb{F}_p)$ has only one lift to $H^*(BN(T_{U(n)});\mathbb{F}_p)$. Using this mod-p information one shows that all triangles

$$\begin{array}{ccc} BP & \longrightarrow & BP' \\ & {\scriptstyle f_P}\searrow \quad \swarrow{\scriptstyle f_{P'}} & \\ & X & \end{array}$$

given by morphisms in $\mathcal{R}_p(U(n))$, commute up to homotopy. This is the trickiest part of the proof and uses Oliver's explicit description of p–stubborn subgroups of $U(n)$ [Ol 1] and a lemma which, for any abelian p–toral group A, calculates the mod–p cohomology of the mapping space $map(BA,X)$ [No 5; Theorem 10.1]. That tells us that the maps f_P define an $\mathcal{R}_p(U(n))$–invariant representation.

Finally, one has to show that this $\mathcal{R}_p(U(n))$–invariant representation extends to a map $f : \underset{\mathcal{R}_p(U(n))}{hocolim}\, EU(n) \times_{U(n)} \mathcal{I} \to X$, which, because X is p–complete, establishes a homotopy equivalence $BU(n)^\wedge_p \to X$. The obstruction groups for this extension are given by higher limits of the functor $\Pi_i^X(U(n)/P) := \pi_i(map(BP,X)_{f_P}$ (Corollary A.3 or Theorem 2.5). Fortunately, the mapping spaces $map(BP,X)_{f_P}$ are computable and there exists a natural equivalence $\Pi_i^X \xrightarrow{\simeq} \Pi_i^{U(n)}$, where $\Pi_i^{U(n)}$ is the functor given by replacing X by $BU(n)^\wedge_p$. As Jackowski, McClure and oliver showed [J–M–O 1], all higher limits of $\Pi_i^{U(n)}$ vanish and so do all obstruction groups involved. This finishes the proof of Theorem 4.3.

We finish this section with

4.4 Conjecture. *Theorem 4.3 holds for every connected compact Lie group.*

5. Lie group theory for finite loop spaces and p–compact groups.

The starting point of this theory was an idea of Rector [Re 1,2], who suggested studying a compact Lie group G (as Lie group) by looking at its classifying space BG and expressing classical Lie group notions in terms of classifying spaces. This would allow Lie group theory to be applied to a much larger class of spaces, namely finite loop spaces.

A loop space $L := (L, BL, e)$ consists of a pair of spaces L and BL, BL pointed, and a homotopy equivalence $e : \Omega BL \simeq L$ defining a loop structure on L. The space BL is called the classifying space of L. A loop space L inherits properties from the space L, e.g. a loop space is called finite, if $H^*(L;\mathbb{Z})$ is a finitely generated as graded abelian group (usually, one asks for an equivalence between L and a finite CW–complex, but the homological condition is sufficient for most of the results about finite loop spaces). Examples of finite loop spaces are given by compact

Lie groups. For every compact Lie group G, there exists a canonical equivalence $e : \Omega BG \simeq G$ which establishes a finite loop space structure (G, BG, e) on G.

Rector gave definitions for subgroups, maximal tori and Weyl groups of a finite loop space[Re 1,2] and used this "Lie group theory" for a study of loop space structures on S^3. In particular, he showed that there exist uncountable many loop structures on S^3 (compare this with Theorem 4.1) and, with the help of McGibbon at the prime 2 [McG], that the property of admitting a maximal torus distinguishes the genuine loop space structure of all the others [Re 1].

The real break through in this theory was by Dwyer and Wilkerson [D–W 5]. Instead of looking at finite loop spaces, they passed to p–adic completions and called a loop space $X := (X, BX, e)$ a p–compact group if X is $\mathbb{F}_p$–finite and if BX is a p–complete space. The latter if–part is equivalent to the condition that X is p–complete and that $\pi_0(X)$ is a finite p–group. Again, the main examples are given by compact Lie groups. But the triple $(G_p^\wedge, BG_p^\wedge, e)$ is only a p–compact group if $\pi_0(G)$ is a finite p–group. As already mentioned in Section 3, further examples are given by pairs $(\Omega BX, BX)$, where BX has polynomial mod–p cohomology.

In contrast to finite loop spaces, Dwyer and Wilkerson showed that every p–compact group has a maximal torus and a Weyl group with similar properties known for classical Lie group theory [D–W 5]. This astonishing similarirty was extended by the same people and by Møller and the author [M–N 1] to the philosophical theorem that p–compact groups enjoy almost every property of compact Lie groups.

If you believe in this similarity , then the game goes as follows: You take your favourite theorem about compact Lie groups, you translate it into the language of classifying spaces or p–compact groups and you try to find a "new" proof in these terms. If you are successful, you get a new and interesting result about classifying spaces or p–compact groups. There is a lack of a notion of the Lie algebra. But, when proving conjectures suggested by classical Lie group theory, the existence of maximal tori and Weyl groups and an induction principle, due to Dwyer and Wilkerson [D–W 6], are a good replacement for the Lie algebra.

Next we set up part of the dictionary (for p–compact groups) and try to explain what the new techniques in the proofs are. Several of the notions have also a straightforward translation into the category of finite loop spaces.

5.1 special p–compact groups: The component X_0 of the unit of a p–compact group X is given by one component of X or by the universal cover of BX. A p–compact torus is a triple (T, BT, e) where $T \simeq K(\mathbb{Z}_p^{\wedge n}, 1)$ is an Eilenberg–MacLane space of degree 1. A p–compact group X is called toral if X_0 is a p–compact torus, finite if X is homotopically discrete, and abelian if $map(BX, BX)_{id} \simeq BX$. For honest abelian compact Lie groups, the last definition is actually a theorem.

5.2 Homomorphisms: A homomorphism $f : X \to Y$ is a pointed map $Bf : BX \to BY$. The homomorphism f is an isomorphism if Bf is a homotopy equivalence. It is a monomorphism if the homotopy fiber Y/X of Bf is $\mathbb{F}_p$–finite or equivalently if $H^*(BX; \mathbb{F}_p)$ is a finitely generated module over $H^*(BY; \mathbb{F}_p)$ ([D–W 5; Prposition 9.11]). This also defines subgroups. These definitions are motivated by the fact that every monomorphism $\rho : G \to H$ of compact Lie groups establishes a fibration $H/G \to BG \to BH$ and by a theorem of Quillen saying that ρ has a finite p–torsion free kernel if and only if $H^*(BG; \mathbb{F}_p)$ is finitely generated over $H^*(BH; \mathbb{F}_p)$.

A short exact sequence $X \to Y \to Z$ of p–compact groups is a fibration $BX \to BY \to BZ$.

Two homomorphisms $f_1, f_2 : X \to Y$ are conjugate if Bf_1 and Bf_2 are freely homotopic. A subgroup $i_1 : X_1 \hookrightarrow Y$ is subconjugate to another subgroup $i_2 : X_2 \hookrightarrow Y$ if there exists a homomorphism $j : X_1 \to X_2$ such that $i_2 j$ and i_1 are conjugate.

5.3 Elements of p–compact groups: An element of a p–compact group X of order p^n is a monomorphism $\mathbb{Z}/p^n \to X$.

5.4 Proposition. [D–W 5; Proposition 5.4] *Every p–compact group X has an element of order p.*

For a proof of such a statement in classical Lie group theory, usually a 1-dimensional parametrized subgroup is constructed with the help of the tangent bundle, i.e. a subgroup isomorphic to S^1. So we need a "new" proof.

Proof of 5.4. For a compact Lie group G the composition $G^{p-1} \cong \{(g_1, ..., g_p) \in G^p | \prod_i g_i = 1\} \to G^p \to G^p/\Delta(G)$ is a homeomorphism and $\mathbb{Z}/p$ equivariant, where $\Delta : G \to G^p$ is the diagonal embedding and where $\mathbb{Z}/p$ acts on G^p via cyclic permutations. The fixed–point set $(G^{p-1})^{\mathbb{Z}/p} \cong (G^p/G)^{\mathbb{Z}/p}$ equals the set of elements of order p.

Now we can argue for p–compact groups. The diagonal $\Delta : X \to X^p$ is a $\mathbb{Z}/p$–equivariant homomorphism and establishes a $\mathbb{Z}/p$–equivariant fibration $X^p/X \to BX \to BX^p$. Taking homotopy fixed points yields a fibration

$$(X^p/X)^{h\mathbb{Z}/p} \to BX^{h\mathbb{Z}/p} \simeq map(B\mathbb{Z}/p, BX) \xrightarrow{\Delta^{h\mathbb{Z}/p}} (BX^p)^{h\mathbb{Z}/p} \simeq BX \ .$$

The equivalence follows from the identities $map(\mathbb{Z}/p, BX) = BX^{\mathbb{Z}/p}$ of $\mathbb{Z}/p$–equivariant spaces and

$$\begin{aligned} BX \simeq & map(E\mathbb{Z}/p, BX) \cong map(E\mathbb{Z}/p \times_{\mathbb{Z}/p} \mathbb{Z}/p, BX) \cong map(E\mathbb{Z}/p \times \mathbb{Z}/p, BX)^{\mathbb{Z}/p} \\ \cong & map(E\mathbb{Z}/p, map(\mathbb{Z}/p, BX))^{\mathbb{Z}/p} \simeq (BX^{\mathbb{Z}/p})^{h\mathbb{Z}/p} \ . \end{aligned}$$

This argument also shows that the map $\Delta^{h\mathbb{Z}/p}$ is given by the evaluation at the basepoint.

Because X is a loop space with $H^*(X; \mathbb{F}_p)$ being finite, the Euler characteristic $\chi(X^{p-1}) = \chi(X^p/X)$ vanishes. Therefore, by Smith theory for homotopy fixed points (Theorem C.3), we have $\chi(X^p/X)^{h\mathbb{Z}/p}) \equiv 0 \bmod p$. The constant map $const : B\mathbb{Z}/p \to BX$ gives rise to one homotopy fixed point of X^p/X (for compact Lie groups this is given by the unit) which belongs to a contractible component as the above fibration shows. Hence, this component has Euler characteristic 1. Thus, there must be another one which gives rise to a nontrivial map $B\mathbb{Z}/p \to BX$. Because of the structure of $H^*(B\mathbb{Z}/p; \mathbb{F}_p)$ this has to be a monomorphism. □

With similar methods Dwyer and Wilkerson showed that, if X is connected, every elemnt of order p^n has a p–th root, i.e. every monomorphism $\mathbb{Z}/p^n \to X$ extends to $\mathbb{Z}/p^{n+1}$. Taking p–th roots up to infinity defines a map $BS^1{}^\wedge_p \simeq (B\mathbb{Z}/p^\infty)^\wedge_p \to BX$, which establishes a monomorphism $S^1{}^\wedge_p \to X$ of p–compact groups.

5.5 Centralizers: For a homomorphism $f : Y \to X$ of p–compact groups we define the centralizer $C_X(f(Y))$ by the equation $BC_X(f(Y)) := map(BY, BX)_{Bf}$.

If Y is abelian, i.e. $C_Y(Y) \cong Y$, the homomorphism f factors over $C_X(f(Y))$. Evaluation induces a map $BY \times BC_X(f(Y)) \to BX$ and therefore a homomorphism $Y \times C_X(f(Y)) \to X$ of p–compact groups. If Y is a p–compact toral group , the centralizer is again a p–compact group [D–W 5; Propositions 5.1 and 6.1]. The motivation for this definition comes from Theorem 2.1, which says that for a homomorphism $\rho : P \to G$ from a p–toral group into a compact Lie group the above defining equation is actually a homotopy equivalence.

For finite loop spaces this definition does not make that much sense, because Theorem 2.1 is not true integrally (see [No 2] or [Za 4]).

5.6 Maximal tori: A monomorphism $T \to X$ of a p–compact torus T into a p–compact group X is a maximal torus if $C_X(T)$ is a p–compact toral group and if $C_X(T)/T$ (T is abelian) is homotopically discrete.

For a finite loop space L, we call a monomorphism $T \to L$ of an honest torus considered as finite loop space into L a maximal torus if $rk(T) = rk(L)$. Here, following a result of Hopf [Ho], the rank is given by the transcendence degree of $H^*(BL, \mathbb{Q})$ over $\mathbb{Q}$. This definition is the original one of Rector [Re 1,2]. It can be pushed forward to p–compact groups by completion and is then equivalent to the above one [M–N 2].

The first definition is motivated by the fact that, for connected compact Lie groups, the maximal torus is self centralizing, and the second by the fact that the rank of a connected compact Lie group, defined as above, equals the dimension of the maximal torus.

5.7 Theorem. [D–W 5; 8.11, 8.13 and 9.1] *Let X be a p–compact group. Then, X has a maximal torus $T_X \to X$ and any two maximal tori are conjugate.*

In general, finite loop spaces do not enjoy this property, as the examples of Rector show [Re 1].

Outline of the proof. Without loss of generality we can assume that X is connected. Then, by 5.3, there exists a monomorphism $S^1 \to X$. If the centralizer $C := C_X(S^1)$ is smaller than X, it has a maximal torus $T \to C$ by induction hypothesis. And the composition $T \to C \to X$ is a maximal torus of X. Here, the size of a p–compact group is given by the cohomological dimension, i.e. the highest degree of a non vanishing mod–p cohomology class, and the number of components. If C and X have the same size, one can show that $C \cong X$, that $S^1 \subset X$ is a central subgroup, that there exists a short exact sequence $S^1 \to X \to \overline{X} := X/S^1$ of p–compact groups and that $\overline{X}$ is smaller than X. By induction hypothesis, there exists a maximal torus $\overline{T} \to \overline{X}$. Because every extension of a torus by a torus is again a torus, a pull back yields a maximal torus of X. The induction starts from finite groups or from p–compact toral groups, for which the first part is obvious.

For a classical proof of the second part, usually the fixed point set $G/T_1)^{T_2}$ is analysed, where $T_1, T_2 \subset G$ are two different maximal tori of G. Every fixed point conjugates T_2 into T_1. By the general philosophy, fixed–points are replaced by homotopy fixed–points and Smith theory is still available. In a little more detail, the pull back diagram

$$\begin{array}{ccccc} X/T_1 & \longrightarrow & E & \longrightarrow & BT_2 \\ \| & & \downarrow & & \downarrow \\ X/T_1 & \longrightarrow & BT_1 & \longrightarrow & BX \end{array}$$

given by two maximal tori $T_1, T_2 \subset X$, establishes a T_2–proxy action on X/T_1. (For proxy actions see Appendix C.) Every homotopy fixed point is a section in the upper row and defines a lift from BT_2 into BT_1 and therefore conjugates T_2 into T_1. The set $(X/T_1)^{hT_2}$ is $\mathbb{F}_p$–finite and for the Euler characteristics we have the identity $\chi((X/T_1)^{hT_2}) = \chi(X/T_1) \neq 0$ (Theorem C.5 and C.6). The inequality is shown in [D–W 5; 9.5] (see also Theorem 5.9). This shows that there exists at least one homotopy fixed point. □

5.8 Weyl spaces and Weyl groups: Let $T_X \to X$ be a maximal torus of a p–compact group X. We think of $BT_X \to BX$ as being a fibration. Then, the Weyl space $\mathcal{W}_X$ is defined to be the space of all fiber maps over the identity. By the arguments in the proof of the second part of Theroem 5.7 we have a proxy action of T_X on X/T_X and an equivalence $\mathcal{W}_X \simeq (X/T_X)^{hT_X}$. This fact was used by Dwyer and Wilkerson to show that $\mathcal{W}_X$ is homotopically discrete and that $W_X := \pi_0(\mathcal{W}_X)$ is a finite group under composition. Because all maximal tori are conjugate, the definition of W_X does not depend essentially on the chosen maximal torus. Dwyer and Wilkerson also proved the following analogues of well known results about compact Lie groups.

5.9 Theorem. [D–W; 9.5 and 9.7] *Let $T_X \to X$ be a maximal torus of a connected p–compact group X of rank n. Then the following holds:*

(1) *The order of W_X is equal to the Euler characteristic of X/T_X.*
(2) *The action of W_X on BT_X induces a faithful representation*

$$W_X \to Gl(H^*(BT_X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}) \cong Gl(n; \mathbb{Q}_p^\wedge)$$

whose image is generated by pseudo reflections, i.e. W_X is a pseudo reflection group.

(3) *The map $H^*(BX; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q} \to (H^*(BT_X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q})^{W_X}$ is an isomorphism.*

One cannot expect that the Weyl group is always generated by honest reflections, as examples of Clark and Ewing show.

The proof of the first part follows from the above equivalence between the Weyl space and the homotopy fixed–point set and because $(X/T_X)^{hT_X}$ and X/T_X have the same Euler characteristic (Theorem C.5 and formula C.6). The second part is a consequence of the third which is the difficult part of the proof.

5.10 Normalizers and p–normalizers of the maximal torus: Again we think of a maximal torus as being a fibration $BT_X \to BX$. The Weyl space $\mathcal{W}_X$ acts on BT_X via fiber maps. This establishes a monoid homomorphism $\mathcal{W}_X \to aut(BT_X)$ where $aut(BT_X)$ denotes the monoid of all self equivalences of BT_X. Passing to classifying spaces establishes a map $B\mathcal{W}_X \to Baut(BT_X)$ which can be thought of as being a classifying map of a fibration $BT_X \to BN(T_X) \to B\mathcal{W}_X$. The total space gives the classifying space of the normalizer $N(T_X)$ of T_X. This construction is nothing but the Borel construction.

In general $BN(T_X)$ is not a p–compact group, because W_X is not a finite p–group. Let $\mathcal{W}_p$ be the union of those components of $\mathcal{W}_X$ corresponding to a p–Sylow subgroup W_p of W_X. The restriction of the above construction to $\mathcal{W}_p$ gives the classifying space of the p–normalizer $N_p(T_X)$, which is a p–compact group. Since the action of $\mathcal{W}_X$ respects the map $BT_X \to BX$, the monomorphism $T_X \to X$ extends to a loop map $N(T_X) \to X$. The restriction $N_p(T_X) \to X$ is a monomorphism and the Euler characteristic $\chi(X/N_p(T_X)$ is coprime to p. [D–W; Proof of 2.3]. A homotopy fixed–point argument, similiar to that in the proof of Theorem 5.7, shows that p–compact toral subgroups of X are subconjugate to $N_p(T_X)$. That is to say that $N_p(T_X)$ is a p–toral Sylow subgroup of X.

This gives the basis of a Lie group theory for p–compact groups. Now we can look at the wide and rich field of classical Lie group theory and try to rediscover it in this spirit through results about p–compact groups. So, let us continue.

5.11 Centers: A subgroup $Z \subset X$ of a p–compact group X is called central, if evaluation induces an isomorphism $C_X(Z) \cong X$ of p–compact groups. A subgroup $Z(X) \subset X$ is called the center of X if it is central and if every central subgroup $Z \subset X$ is subconjugate to $Z(X)$. That is to say the center is the maximal central subgroup. This already gives an idea how the center can be constructed. The "union" of two central subgroups should be again a central subgroup. The following theorem was proved independently by Dwyer and Wilkerson and by Møller and the author.

5.12 Theorem. [D–W 6] [M–N 1] *Every p–compact group X has a center $Z(X) \subset X$. There exists a short exact sequence*

$$Z(X) \to X \to \overline{X} := X/Z(X) \ .$$

If X is connected, then $\overline{X}$ is centerfree.

5.13 An induction principle: In [Dw], Dwyer showed that a transfer for "nice" cohomology theories (including mod–p cohomology) exists if the fiber satisfies some finiteness conditions expressed in terms of the cohomology theory. In general, these conditions are slightly weaker as being equivalent to a finite CW–complex. For example, for mod–p cohomology, the fiber only has to be $\mathbb{F}_p$–finite. Applying this to the homomorphism $N_p(T_X) \to X$ of the p–normalizer of the maximal torus of a p–compact group X, Dwyer and Wilkerson showed that $H^*(BX;\mathbb{F}_p)$ is a noetherian algebra and that the map $H^*(BX;\mathbb{F}_p) \to H^*(BN_p(T_X);\mathbb{F}_p)$ satisfies the assumption of Theorem 1.4. That is to say there exists an algebraic decomposition of BX via centralizers of elementary abelian subgroups which can be realized on the geometric level. Let $\mathcal{A}_p(X)$ denote the Quillen category of X, omitting the trivial subgroup. This makes perfect sense, because we already translated all necessary notions, involved in the definition, into the language of p–compact groups. By Lannes' theory, the Quillen category is equivalent to the Rector category of $H^*(BX;\mathbb{F}_p)$. For elementary abelian p–subgroups $V \subset X$, the maps $BC_X(V) \to BX$ induced by evaluation at basepoints are compatible with the morphisms of $\mathcal{A}_p(X)$. Hence the following theorem is a consequence of Theorem 1.4 and Theorem B.2. It also generalizes the decomposition of classifying spaces of compact Lie groups via centralizers of elementary abelain subgroups by Jackowski and McClure (Theorem 1.3)

5.14 Theorem. *The natural map $\operatorname{hocolim}_{\mathcal{A}_p(X)} BC_X(V) \to BX$ is a homotopy equivalence.*

This is the key for the following induction principle of Dwyer and Wilkerson.

5.15 Proposition. *Let $\mathcal{C}l$ be a class of p–compact groups satisfying the following conditions:*

(1) *If $X \in \mathcal{C}l$ and $Y \cong X$, then $Y \in \mathcal{C}l$.*
(2) *The trivial group belongs to $\mathcal{C}l$.*
(3) *If the component X_0 of the unit is contained in $\mathcal{C}l$, then $X \in \mathcal{C}l$.*
(4) *If X is connected and if $X/Z(X) \in \mathcal{C}l$, then $X \in \mathcal{C}l$.*
(5) *If X is connected and centerfree and $Y \in \mathcal{C}l$ for every p–compact group Y with smaller cohomological dimension than X, then $X \in \mathcal{C}l$.*

Then, the class $\mathcal{C}l$ contains every p–compact group.

The proof is nothing but the observation that for a centerfree connected p–compact group X, the centralizer of any subgroup has smaller cohomological dimension than X. The cohomological dimesion is defined to be the maximal degree of the nonvanishing mod–p cohomology classes.

To demonstrate the induction principle we want to prove the Sullivan conjecture for p–compact groups.

5.16 Theorem. [D–W 5] *Let X be a p–compact group, and let K be a p–complete $\mathbb{F}_p$–finite space. Then evaluation induces an equivalence $ev : map(BX, K) \xrightarrow{\simeq} K$.*

Proof. Let $\mathcal{C}l$ denote the class of all p–compact groups which satisfy the statement. Then, the first two conditions are obviously satisfied.

Any p–compact group X fits into a short exact sequence $X_0 \to X \to \pi := \pi_0(X)$ of p–compact groups. If $X_0 \in \mathcal{C}l$, one can show that $map((B\pi, K) \to map(BX, K)$ is an equivalence. Hence, the third condition is satisfied by the Sullivan conjecture for finite groups [Mil].

For any connected p–compact group X, Theorem 5.12 establishes a fibration $BZ(X) \to BX \to B\overline{X} := B(X/Z(X))$. Actually, as for compact Lie groups, this is a principal fibration with classifying map $B\overline{X} \to B^2(Z(X))$. The center is a product of a p–compact torus and a finite abelian p–group and therefore satisfies the Sullivan conjecture. In this situation we can apply a lemma of Zabrodsky [Za 3] to show that $map(B\overline{X}, K) \to map(BX, K)$ is an equivalence. Hence, the fourth condition is satisfied.

To prove the fifth condition, we use the decomposition theorem (Theorem 5.14). Let X be connected and centerfree. We have

$$map(BX, K) \simeq map(\underset{\mathcal{A}_p(X)}{hocolim}\, BC_X(V), K)\ .$$

The centralizers are smaller than X. Thus if they satisfy the theorem, the higher limits in the spectral sequence of Theorem A.2 for calculating the homotopy groups of the latter mapping space have to be taken over the constant functors with $\pi_*(K)$ as value. But Theorem 1.4 implies that all higher limits of a constant functor on $\mathcal{A}_p(X)$ vanish. Hence, we have $map(BX, K) \simeq K$ and $X \in \mathcal{C}l$. □

This induction principle is quite a powerful tool. For example, along the same lines, Møller proved that every homomorphism $f : X \to Y$ between connected p–compact groups is trivial if and only if the restriction to a maximal torus is trivial if and only if $H^*(Bf; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is trivial [Mø3]. And Dwyer and Wilkerson showed that there exists an equivalence $BZ(X) \simeq map(BX, BX)_{id}$ for any p–compact group X. The map is an adjoint of a multiplication $Z(X) \times X \to X$

on the level of classifying spaces [D–W 6]. This generalizes parts of the results of Jackowski, McClure and Oliver (Theorem 2.7 for connected compact Lie groups). This equivalence will become an important fact in the further study of p–compact groups; in particular in the analysis of homomorphisms between p–compact groups and proofs of homotopy uniqueness properties using the decomposition of BX via centralizers of elementary abelian subgroups. For example, the equivalence allows it to be shown that $BZ(C_X(V)) \to map(BC_X(V), BX)_{Bi}$ is an equivalence for any elementary abelian subgroup $V \subset X$.

The final goal of the theory of p–compact groups is a complete classification. Again, classical Lie group theory serves as a guide. On the one hand every connected compact Lie group has a finite cover which is a product of a simply connected compact Lie group and a torus and every simply connected compact Lie group splits into a product of simple simply connected pieces. On the other hand, two connected compact Lie groups are isomorphic if the normalizers of the maximal tori are. (This is implicitly contained in Bourbaki [Bour]. For an explicit proof see [C–W–W] (only for semi simple Lie groups), [Mø2], [Os] or [No 6]). The first statements can completely reproved for p–compact groups due to work of Dwyer and Wilkerson [D–W 7] and Møller and the author [M–N 1] [No 8]. For the second statement there exist partial results of the latter group of authors [M–N 2].

5.17 Theorem. [M–N 1] *Let X be a connected p–compact group. Then there exists a short exact sequence*

$$K \to X_s \times T \to X$$

of p–compact groups, where K is a finite group, where T is a p–compact torus and where X_s is a simply connected p–compact group. The group K is a central subgroup of $X_s \times T$.

The simply connected part X_s is given by the 2–connected cover of BX which also is a p–compact group. The toral part T is given by the component of the unit of the center of X. Because $C_X(T) \cong X$, there exists a map $X_s \times T \to X$. To show that this is a finite covering comes down to a proof that the center of a p–compact group is finite if and only if the fundamental group is finite, a well known fact for connected compact Lie groups.

For every simple connected compact Lie group G, the associated representation $W_G \to Gl(H^2(BT_G; \mathbb{Q})$, is irreducible. This property is used for the definition of a simple p–compact group, i.e. X is simple if the associated representation $W_X \to Gl(H^2(BT_X; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is irreducible.

5.18 Theorem. [D–W 7] [No 8] *Let X be a simply connected p–compact group. Then there exists a splitting $X \cong \prod_i X_i$ into a product of simple simply connected p–compact groups X_i.*

Dwyer and Wilkerson proved this for all primes and independently the author for odd primes. Actually, the statement is first proved for centerfree p–compact groups and then for simply connected p–compact group by passing to the universal cover. For centerfree p–compact groups the integral representation $W_X \to Gl(H^2(BT_X; \mathbb{Z}_p^\wedge)$ is under control and splits into a direct sum, where each piece belongs to an irreducible factor of the associated rational representation of W_X [D–W 7] [No 7]. This splitting gives rise to a splitting $W_X \cong \prod_i W_i$ of W_X

and $T_X \cong \prod T_i$ such that W_i acts only on T_i nontrivially. The centralizer $C_X(T_i)$ for fixed i splits into a product $X_i \times \prod_{i \neq j} T_j$. This part of the proof goes the same way as for simply connected compact Lie groups. The construction of a homomorphism $\prod_i X_i \to X$ is the difficult part of the proof of Theorem 5.18. Basically, the centralizer $C_X(X_i)$ has to be computed.

Theorem 5.9 connects connected p–compact groups with pseudo reflection groups. The list of Clark and Ewing gives a complete classification of such irreducible gadgets. By the above two theorems, a complete classification of connected p–compact groups consists of the construction of a simple simply connected p–compact group and a homotopy uniqueness result for each irreducible pseudo reflection group of the list. For most of the irreducible pseudo reflection groups examples are constructed (see section 3). Homotopy uniqueness results, in particular if p–torsion in the cohomology is around, are the main missing link for a complete classification of connected p–compact groups.

The most general homotopy uniqueness results in terms of normalizers of maximal tori so far are proved by Møller and the author. We say that two p–compact groups X and Y have the same (p–adic) Weyl group type if X and Y have the same rank n and if the two associated integral representations $W_X, W_Y \to Gl(n, \mathbb{Z}_p^\wedge)$ are equivalent.

5.19 Theorem. [M–N 2] *Let p be an odd prime. Let G be a connected compact Lie group such that $H^*(G;\mathbb{Z})$ has no p–torsion. Let X be a connected p–compact group with the same p–adic Weyl group type as G. Then X and $G_p^\wedge$ are isomorphic as p–compact groups.*

The proof is based on Theorem 4.3, which states a homotopy uniqueness result based on mod–p cohomology. The main part of the proof is to show that the p–compact groups under consideration have torsion free p-adic cohomology. Again, this is first proved for unitary groups and then extended to the other cases. For $p = 2$ a similar result is true for quotients of products of unitary and special unitary groups, but again, $SU(2)$ is excluded.

For $p = 2$ the Weyl group data are not sufficient to distinguish between connected p–compact groups as a comparison of $SO(2n+1)$ and $Sp(n)$ shows. In general the following conjecture should be true. As usual, it generalizes a known statement about connected compact Lie groups.

5.20 Conjecture. *Two connected p–compact groups X and Y are isomorphic if and only if the normalizers of the maximal tori are isomorphic (as loop saces). At odd primes the normalizer splits and the Weyl group data are sufficient to distinguish between connected p–compact groups.*

There has also been some work done on the analysis of endomorphisms of p–compact groups. Møller studied rational self equivalences. These are endomorphisms $f : X \to X$ such that $H^*(Bf; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is an isomorphism. He was able to generalize parts of the Jackowski–McClure–Oliver theorem (Theorem 2.7) and reduced the homotopy clasification of rational self equivalences to the case of endomorphisms of simple simply connected p–compact groups [Mø 4,5]. For endomorphisms of nonconnected p–compact groups, he proved similar results to those he proved for nonconnected compact Lie groups (see Section 2).

6. Finite loop spaces and integral questions.

As we already mentioned in the previous section, Rector suggested studying compact Lie groups from the homotopy point of view, i.e. passing to classifying spaces [Re 1,2]. If we consider a compact Lie group just as a topological space, we would lose too much information. Although the homotopy type of G distinguishes between two simple connected compact Lie groups [B–B], this is not true in general, even not for semi simple groups. Counterexamples may also be found in [B–B]. The concept of Rector is based on the hope that the classifying space BG contains all the information about the compact Lie group G. This was independently proved by Møller [Mø2] and Osse [Os] for connected compact Lie groups and in general by the author [No 6].

6.1 Theorem. [Mø2], [Os] [No 6] *Two compact Lie groups G and H are isomorphic if and only if the classifying spaces BG and BH are homotopy equivalent.*

The p–adic completion of a connected finite loop space gives a p–compact group, and the rationalisation is a product of rational Eilenberg–MacLane spaces. Hence, via Sullivan's arithmetic square [Su], finite loop spaces are built out of p–compact groups. This gives a way to transport p–compact group results to theorems about finite loop spaces.

Most of the recent research on finite loop spaces can be understood as an attempt to solve the following conjecture, which we could trace back to Wilkerson [Wi 2].

6.2 Conjecture. [Wi 2] *Every finite loop space with maximal torus is equivalent to a compact Lie group (as finite loop space).*

In a first approach, finite loop spaces L were considered whose classifying space BL has the same (adic) genus as BG for a given connected compact Lie group G. That is that all p–adic completions $BG^\wedge_p$ and $BL^\wedge_p$ are homotopy equivalent. Because BG as well as BL is rationally a product of Eilenberg–MacLane spaces, this also implies that the rationalizations are equivalent. Also, by passing to loops, the space L has the same genus as G, if G is connected. Actually, the genus of a space was defined via localization, but using completion is more appropriate to this kind of question. In particular, if BL is in the genus of BG, we can think of BL as built out of the p–adic completions of BG, the rationalisation of BG and some glueing data, encoded in a self map of the adele type of BG. Actually, the adele type of BG is a product of Eilenberg–MacLane spaces. Hence, the glueing code is completely contained in cohomological information. If, for a finite loop space L, the classifying space BL has the same genus as BG, we say that L is a fake Lie group of type G.

Rector solved the above conjecture for S^3 under this genus assumption by showing (with a little help from McGibbon at the prime 2 [McG]) that the property of admitting a maximal torus distinguishes the genuine group among the fakes [Re 1]. Actually, because of the homotopy uniqueness property of $BS^{3\wedge}_p$ (Theorem 4.1), every loop space structure on S^3 is a fake Lie group of type S^3. The genus assumption is superflous in this case.

Rector's result was extended by Smith and the author to the case of simply connected compact Lie groups [N–S 1,2,3] [No 3]. In a series of papers, they analysed the genus of a classifying space of a connected compact Lie group G and arrived at the following result:

6.3 Theorem. [N–S 1,2,3] [No 3] *Let G be a connected compact Lie group and L a fake Lie group of type G.*

(1) *If L admits a maximal torus $T_L \to L$, then there exists a connected compact Lie group H such that L and H are isomorphic as finite loop spaces.*
(2) *If G is simply connected in addition, and L admits a maximal torus, then L and G are isomorphic.*

This solves the above conjecture under the genus assumption. The proof is based on an identification of several Weyl groups connected with L, the Weyl group of L as finite loop space, of $L_p^\wedge$ as p–compact group (for all primes) and the rational Weyl group of L, which can be defined to be the Galois group of the integral ring extension $H^*(BL;\mathbb{Q}) \to H^*(BT_L;\mathbb{Q})$. Moreover, even together with the action on the maximal torus, all these Weyl groups turn out to be isomorphic to W_G. These identifications allow the glueing data for BL to be decoded out of the existence of a maximal torus.

By a theorem of Atiyah [At 2], complex K–theory together with the Adams operations determines the mod–p cohomology of simply connected spaces as algebras over the Steenrod algebra, if the integral cohomology is torsionfree. The Chern character connects complex K–theory with rational cohomology, and complex K–theory determines the rational cohomology ring. Hence, for nice spaces, complex K–theory contains all information about the glueing data of an arithmetic square. It has turned out to be quite a useful tool in the theory of finite loop spaces. For example, a fake Lie group L of type G has a maximal torus if and only if $K(BG) \cong K(BL)$ as λ–rings [N–S 1], and two fake Lie groups L_1 and L_2 of the same type are isomorphic if $K(BL_1) \cong K(BL_2)$ as λ–rings [No 4].

The following theorem is a consequence of the mod–p homotopy uniqueness properties of connected compact Lie groups (Theorem 4.3).

6.4 Theorem. [No 5] *Let G be a quotient of a product of unitary and special unitary groups, different from $SU(2)$, such that $H^*(G;\mathbb{Z})$ is torsionfree. Let X be simply connected CW–complex of finite type such that $H^*(X;\mathbb{Z})$ is torsionfree. Then, BG and X are homotopy equivalent if and only if $K(BG) \cong K(X)$ as λ–rings.*

By Atiyah's result it follows that BG and X have isomorphic mod–p cohomology, and by Theorem 4.3 it follows that the p–adic completions of BG and X are homotoy equivalent. The rationalizations are equivalent because both have isomorphic complex K–theory. Hence, X has the same genus as BG. In fact both spaces are equivalent because of the K–theory isomorphism. This outlines the proof of Theorem 6.4.

Recently, Møller and the author used the ideas of the proof of Theorem 6.3 to study the Weyl group of a general connected finite loop space with maximal torus [M–N 2]. They showed that at least the Weyl group action is the expected one.

6.5 Theorem. [M–N 2] *Let L be a finite loop space with maximal torus $T_L \to L$. Then, we have $H^*(BL;\mathbb{Q}) \cong H^*(BT_L;\mathbb{Q})^{W_L}$ and the the representation $W_L \to Gl(H^2(BT_L;\mathbb{Q})$ is faithful and represents W_L as a crystallographic group.*

This is an integral version of Theorem 5.9, which the proof is very much based on. It is also one step forward to a proof of conjecture 6.2. To complete the proof, a complete classification of connected p–compact groups is necessary in terms of the Weyl group action on the maximal torus or in terms of the normalizer of the maximal torus.

A. Homotopy colimits.

In this section we recall a construction for homotopy colimits which goes back to Segal [Se] and some spectral sequences related to this construction. Let $\mathcal{C}$ be a small (topological) category (as defined in [Se]), and let $F : \mathcal{C} \to \mathcal{T}op$ be a covariant (continous) functor into the category of (compactly generated) topological spaces. The homotopy colimit $\underset{\mathcal{C}}{hocolim}\, F$ can be thought of as a kind of bar construction. It can be constructed as the quotient space

$$\underset{\mathcal{C}}{hocolim}\, F := (\coprod_{n\geq 0} \coprod_{c_0\to\ldots\to c_n} F(c_0) \times \Delta^n)/\sim$$

where c_i is an object of $\mathcal{C}$, where Δ^n is the n–simplex and where each face or degeneracy map between the sequences $c_0\to\ldots\to c_n$ gives rise to the obvious identification. This construction is obviously functoral with respect to (continous) natural transformations. The natural transformation from F to the constant functor $* : \mathcal{C} \to \mathcal{T}op$ taking a point as value induces a map $\underset{\mathcal{C}}{hocolim}\, F \to \underset{\mathcal{C}}{hocolim} * =: B\mathcal{C}$. The target is called the classifying space of the category $\mathcal{C}$. In the above construction this map is given by the projection onto the second factor. The above construction also allows an obvious filtration of the homotopy colimit given by taking the coproduct for $0 \leq n \leq N$. This filtration gives rise to a first quadrant cohomological spectral sequence which calculates the cohomology of the homotopy colimit. The E^2–term is formed by higher derived functors of the inverse limit functor of the contravariant functors $H^q(F(-)) : \mathcal{C} \to \mathcal{A}b$ into the category of abelian groups, established by taking cohomology groups of $F(c)$.

A.1 Theorem. *For any covariant functor $F : \mathcal{C} \to \mathcal{T}op$ there exists a spectral sequence*

$$E^2_{p,q} = \underset{\overleftarrow{\mathcal{C}}}{\lim}{}^p\ H^q(F(-)) \Longrightarrow H^{p+q}(\underset{\mathcal{C}}{hocolim}\, F)\ .$$

For a proof see [B–K], where a different approach for the construction of the homotopy colimit is used (see also [Se]).

Important examples of the homotopy colimit construction are given by:

a) **Pushout diagrams**: Let $\mathcal{C}$ be the category $\mathcal{C} := \{c_1 \longleftarrow c_0 \longrightarrow c_2\}$. Then, for any functor $F : \mathcal{C} \to \mathcal{T}op$, the homotopy colimit is homotopy equivalent to the pushout of the diagram $F(c_1)\longleftarrow F(c_0)\longrightarrow F(c_2)$. The spectral sequence reduces to the Mayer–Vietoris sequence.

b) **Mapping telescopes**: let $\mathcal{C} := \mathbb{N}$ be the category given by the totally ordered set of the natural numbers. Then, for any functor $F : \mathbb{N} \to \mathcal{T}op$, the homotopy colimit is equivalent to the mapping telescope of the sequence

$$F(0) \to F(1) \to \ldots \to F(n) \to \ldots\ .$$

The spectral sequence reduces to the Milnor sequence for the cohomology of mapping telescopes.

c) **Borel constructions**: To any topological group G we can associate a category βG with one object and whose endomorphisms are given by G. In this case, a functor $F : \beta G \to \mathcal{T}op$ is nothing but a G–space X. The homotopy colimit of F is equivalent to the Borel construction $EG \times_G X$. If X is a point, then $\underset{\mathcal{C}}{hocolim}\, F \simeq EG/G = BG$ is just the classifying space of the group G (for details see [Se]).

Let $\mathcal{C}$ be a discrete category, and let $F : \mathcal{C} \to \mathcal{T}op$ be a functor. For any space X, the filtration of the homotopy colimit $\underset{\mathcal{C}}{hocolim}\, F$ establishes a tower of fibrations under $map(\underset{\mathcal{C}}{hocolim}, X)$. This tower gives rise to a spectral sequence calculating the homotopy groups of $map(\underset{\mathcal{C}}{hocolim}, X)$. In more detail, the restrictions to $F(c)$ for each $c \in \mathcal{C}$ establish a map

$$R : [\underset{\mathcal{C}}{hocolim}\, F, X] \to \underset{\overleftarrow{\mathcal{C}}}{\lim}[F(-), X] \ .$$

Let $\hat{f} = (f_c)_{c\in\mathcal{C}} \in \underset{\overleftarrow{\mathcal{C}}}{\lim}[F(-), X]$. Let $\phi_n : \mathcal{C} \to \mathcal{A}b$ be the contravariant functor given by $\phi_n(c) := \pi_n(map(F(c), X)_{f_c}$.

A.2 Theorem. *Under the same assumptiosn as above there exists a spectral sequence*

$$E_2^{p,q} = \underset{\overleftarrow{\mathcal{C}}}{\lim}{}^p \ \phi_q \Longrightarrow \pi_{q-p}(map(\underset{F}{hocolim}, X)_{R^{-1}(\hat{f})}) \ ,$$

which converges strongly if $\underset{\overleftarrow{\mathcal{C}}}{\lim}{}^p \phi_q = 0$ *for all* $p \geq N$ *and all* q.

Here, $map(\underset{F}{hocolim}, X)_{R^{-1}(\hat{f})})$ means the union of all components of maps $f : BG \to X$ such that $R(f) = \hat{f}$. For a proof see [B–K] and [Wo 1]. Bousfield and Kan arrived at this result by contructing a spectral sequence for homotopy inverse limits and Woijtkowiak discusses carefully all questions related to choosing basepoints and what happens when the fundamental group is nonabelian.

If we are only interested in the set of components of the mapping space we have the following corollary.

A.3 Corollary. *Under the above assumptions, the set* $R^{-1}(\hat{f})$ *is non empty if* $\underset{\overleftarrow{\mathcal{C}}}{\lim}{}^{n+1} \ \phi_n = 0$ *for all* $n \geq 1$*, and contains at most one element if* $\underset{\overleftarrow{\mathcal{C}}}{\lim}{}^{n} \ \phi_n = 0$ *for all* $n \geq 1$.

B. Lannes' Theory.

In this section we recall some results and basic definitions of Lannes' theory. Proofs and ideas of proofs are completely omitted. The material is taken from [La 1,2].

For the following we fixed a prime p. The cohomology groups are always taken with coefficients in $\mathbb{Z}/p$ and $H^*(\ )$ always means $H^*(\ ;\mathbb{Z}/p)$. We denote by $\mathcal{K}$ the category of unstable algebras over the Steenrod algebra $\mathcal{A}_p$.

Let V be an elementary abelian p–group. An algebra A over $\mathcal{A}_p$ is called of finite type if A is finite in each dimension.

B.1 Theorem. [La 2] *If X is a p–complete space and $H^*(X)$ is of finite type, then the canonical map*

$$[BV, X] \longrightarrow Hom_{\mathcal{A}_p}(H^*(X), H^*(BV))$$

is an isomorphism.

The evaluation map

$$BV \times map(BV, X) \longrightarrow X$$

induces a cohomological map

$$H^*(X) \longrightarrow H^*(BV) \otimes H^*(map(BV, X)) \ .$$

Lannes studied the functor $T_V : \mathcal{K} \longrightarrow \mathcal{K}$ which is the left adjoint of the functor $H^*(BV) \otimes_{\mathbb{Z}/p} -$. Taking the adjoint of the evaluation map yields a map

$$T_V H^*(X) \longrightarrow H^*(map(BV, X)) \ .$$

For any map $g : BV \longrightarrow X$, there is an associated direct summand $T_V(H^*(X), g^*)$ of $T^V H^*(X)$ which corresponds to the summand $H^*(map(BV, X)_g)$ of $H^*(map(BV, X))$. With respect to this splitting the above map is a direct sum of maps with coordinates

$$T_V(H^*(X), g^*) \longrightarrow H^*(map(BV, X)_g) \ .$$

B.2 Theorem. [La 2] *Let X be a space, such that $H^*(X)$ is of finite type. Let $g : BV \longrightarrow X$ be a map. The map*

$$T^V(H^*(X), g^*) \longrightarrow H^*(map(BV, X^\wedge_p)_g)$$

is an isomorphism if $T^V(H^*(X), g^*)$ is of finite type and one of the following three conditions is satisfied:

1. *$T^V(H^*(X), g^*)$ is zero in degree 1.*
2. *$map(BV, X^\wedge_p)_g$ is 1–connected.*
3. *There is a connected space Z with the property that $H^*(Z)$ is of finite type and a map*

   $$Z \longrightarrow map(BV, X)_g \ ,$$

   such that the associated map

   $$T^V(H^*(X), g^*) \longrightarrow H^*(Z)$$

   is an isomorphism.

B.3 Theorem. [La 2, 3.4.3] *In addition to the assumptions of Theorem B.2, let X be p–complete. Then the following conditions are equivalent:*

1. *$map(BV, X)_g$ is p–complete.*
2. *$T^V(H^*(X), g^*) \longrightarrow H^*(map(BV, X)_g)$ is an isomorphism.*

If we consider a collection S of maps $BV \longrightarrow X$, then of course we get a direct summand $T_V(H^*(X), S^*)$, where S^* is the collection of the associated cohomological maps. The theorems B.2 and B.3 are still true in this situation [La 2].

C. Homotopy fixed–points and Smith–theory.

Let G be a group acting on a space X. Then the fixed–point set can be described as the mapping space $X^G = map_G(*, X)$ of G–equivariant maps from a point into X. The homotopy fixed–point set is defined as the mapping space $X^{hG} := map_G(EG, X)$, where EG is a contractible free G–space. The G–equivariant projection $EG \to *$ induces a map $X^G \to X^{hG}$. As one easily sees, the homotopy fixed–point is equivalent to the space of sections $\Gamma(X_{hG} \to BG)$ of the bundle $X_{hG} \to BG$ given by the Borel construction $X_{hG} := EG \times_G X$. Compared with fixed–point sets, homotopy fixed–point sets have the very nice advantage of only depending on the homotopy type of X; i.e. any G–equivariant map $X \to Y$, which is also a homotopy equivalence (non equivariant) induces an equivalence $X^{hG} \to Y^{hG}$. This follows straight forward from the description in terms of section spaces. In general this is not true for fixed–point sets. But for G a finite p–group, the generalized Sullivan conjecture says that there is a close connection between fixed–point and homotopy fixed–point sets. By functoriality there is a natural map $X^{G\wedge}_p \to X^{\wedge hG}_p$.

C.1 Theorem. *(*[D–M–N] [Ca] [La1,2]*) Let G be a finite p–group acting on a finite G–complex. Then the map $X^{G\wedge}_p \to X^{\wedge hG}_p$ is a weak equivalence.*

In the case of $G = V$ an elementary abelian p–group, there are two ways to get a hold of the cohomology of the homotopy fixed–point set. The first is an output of Lannes' theory and his T–functor. The second is a functoral recipe of calculating the cohomology of X^{hV} on the lines of the localization theorem, a reformulation of classical Smith–theory (e.g. see [Hs]).

For a space X with a V–action there exists a fibration

$$(**) \qquad X^{hV} \simeq \Gamma(X_{hV} \to BV) \to map(BV, X_{hV})_{sec} \to map(BV, BV)_{id}$$

where the total space is the union of the components given by sections in the fibration $X_{hV} \to BV$. Because $map(BV, BV)_{id} \simeq BV$ [Hur] (see the introduction), composition defines a map $X^{hV} \times BV \to map(BV, X_{hV})_{sec}$ which establishes a fiber homotopy trivialization of $(**)$. This was used by Lannes to establish a functor

$$Fix_V(X) := T_V(H^*_V(X; \mathbb{F}_p), sec^*) \otimes_{H^*V} \mathbb{F}_p \ .$$

Here, sec^* denotes the set of all sections on the cohomological level, and $H^*_V(X) := H^*(X_{hV})$ denotes equivariant cohomology. We also defined $H^*_V := H^*(BV; \mathbb{F}_p)$. As in the case of the T–functor, there is a natural map

$$Fix_V(X) \to H^*(X^{hV}; \mathbb{F}_p)$$

which turns out to be an isomorphism in several cases.

C.2 Theorem. [La 2] *Let V act on a p–complete space X whose cohomology $H^*(X; \mathbb{F}_p)$ is of finite type. Then, the natural map*

$$Fix_V(X) := T_V(H^*_V(X; \mathbb{F}_p), sec^*) \otimes_{H^*V} \mathbb{F}_p \to H^*(X^{hV}; \mathbb{F}_p)$$

is an equivalence if and only if the homotopy fixed–point set X^{hV} is p–complete.

Analogously as in Theorem B.2, there are several other conditions which ensure that $Fix_V(X) \to H^*(X^{hT}; \mathbb{F}_p)$ is an isomorphism (see [La 2]). The condition of

X^{hV} being p–complete seems hard to check, but fortunately in all cases relevant for the proofs of the results in the previous sections, it turns out that this condition is always satisfied.

For the following we assume that X is $\mathbb{F}_p$–finite. Let $S \subset H^*(BV;\mathbb{F}_p)$ be the multiplicative subset of all elements of strictly positive degree in the image of $H^*(BV;\mathbb{Z}) \to H^*(BV;\mathbb{F}_p)$. This is isomorphic to the submodule of positive elements of the torsionfree quotient of H^*_V considered as a module over itself. For a nice action of V on X, the above mentioned localization theorem reads

$$S^{-1}H^*(X^V;\mathbb{F}_p) \cong S^{-1}H^*(X_{hV};\mathbb{F}_p) \ .$$

The localized algebra $S^{-1}H^*(X_{hV};\mathbb{F}_p)$ is still an algebra over the Steenrod algebra but may not satisfiy the unstability conditions. Let $Un(S^{-1}H^*(X_{hV};\mathbb{F}_p)$ be the unstable part of $S^{-1}H^*(X_{hV};\mathbb{F}_p)$. Dwyer and Wilkerson proved that

$$H^*_V \otimes Fix_V(X) \cong Un(S^{-1}H^*(X_{hV};\mathbb{F}_p)$$

[D–W 2]. Because $H^*(X_{hV};\mathbb{F}_p)$ is a finitely generated module over H^*_V as a Serre spectral sequence argument shows, the $\mathbb{F}_p$–module $Fix_V(X)$ is also finite. Using Theorem C.2 in addition, the same authors proved

C.3 Theorem. [D–W 5] *Let $V = \mathbb{Z}/p$ be the cyclic group of order p acting on a $\mathbb{F}_p$–finite p–complete space X. If the homotopy fixed–point set X^{hV} is also p–complete, then the following holds:*

(1) *The homotopy fixed point set X^{hV} and the pair $(X_{hV}, BV \times X^{hV})$ are $\mathbb{F}_p$–finite.*
(2) *For the Euler characteristics we have $\chi(X) \equiv \chi(X^{hV})$ mod p.*

This parallels the classical facts that, for $V = \mathbb{Z}/p$ acting nicely on a finite V–complex X, the analogous statements for the actual fixed–point set are true.

The generalization of these results to the case of finite p–groups goes by an induction over the group order. One uses the fact that any finite p–group π fits into a short exact sequence $1 \to \pi_0 \to \pi \to \mathbb{Z}/p \to 1$, that $\mathbb{Z}/p$ acts on the homotopy fixed–point set $X^{h\pi_0} \simeq map_{\pi_0}(E\pi, X)$ and that $X^{h\pi} \simeq (X^{h\pi_0})^{h\mathbb{Z}/p}$. Assuming that for any subgroup $\pi' \subset \pi$ the homotopy–fixed point set $X^{h\pi'}$ is p–complete gives the desired Euler characteristic formula

$$\chi(X) \equiv \chi(X^{h\pi}) \text{ mod } p \ . \tag{C.4}$$

The arguments involved in the proof of Theorem C.3 do not care about the explicit action of the finite p group π on the space X. The fibration $X \to X_{h\pi} \to B\pi$ contains all the necessary information. For example, the homotopy fixed point set $X^{h\pi}$ is equivalent to the space of sections in the above bundle. This motivates the following definition of Dwyer and Wilkerson. A proxy action of π on X is a fibration $X \to E \to B\pi$. The homotopy fixed–point set $X^{h\pi}$ is given by the space of sections of this fibration. Then, Theorem C.3 is still true for proxy actions of finite p groups on $\mathbb{F}_p$–finite spaces.

Using the mod–p approximation $B\mathbb{Z}/p^{\infty\wedge}_{\ p} \to BS^{1\wedge}_{\ p}$ of a torus, Dwyer and Wilkerson extended Theorem C.3 to proxy actions of tori.

C.5 Theorem. [D–W 5] *Let* $X \to E \to BT$ *be a proxy action of a torus on a* $\mathbb{F}_p$*–finite space* X*. Assume that for every finite* p*–subgroup* $\pi \subset T$ *the homotopy fixed–point set* $X^{h\pi}$ *is* p*–complete. Then, for every finite* p*–subgroup* $\pi \subset T$*, the space* $X^{h\pi}$ *is* $\mathbb{F}_p$*–finite and* $\chi(X^{h\pi}) = \chi(X)$*.*

In general it is not known if X^{hT} is $\mathbb{F}_p$–finite or if $\chi(X^{hT}) = \chi)X)$. But in all cases, appearing in the proofs of the above theorems, the homotopy fixed–point set X^{hT} is equivalent to $X^{h\pi}$ for $\pi \subset T$ big enough. This establishes the formula

$$\chi(X^{hT}) = \chi(X) \ . \tag{C.6}$$

References.

[Ad] J.F. Adams, *Maps between classifying spaces II,*, Inventiones Math. **49** (1978), 1-65.

[A–M 1] J.F. Adams & Z. Mahmud, *Maps between classifying spaces*, Inventiones Math. **35** (1976), 1-41.

[A–M 2] J.F. Adams & Z.Mahmud, *Maps between classifying spaces III*, Topological topics, Cambridg Univ. Press (1983), 95–143.

[A–W] J.F. Adams & C.W. Wilkerson, *Finite H–spaces and algebras over the Steenrod algebra*, Ann. Math. **111** (1980), 95-143.

[A–Wo] J.F. Adams & Z. Woijtkowiak, *Maps between p–completed classifying spaces*, Proc. Royal Soc Edinburgh **112 A** (1989), 231-235.

[Ag] J. Aguadé, *Constructing modular classifying spaces*, Isreal J. Math. **66** (1989), 23-40.

[At 1] M.F. Atiyah, *Characters and cohomology of finite groups*, Publ. I.H.E.S. **9** (1961), 23-64.

[At 2] M.F. Atiyah, *Power operations in K–theory*, Q. J. Math. Oxford II **17** (1966), 165-193.

[A–S] M.F. Atiyah, *Equivariant K–theory and completion*, J. Diff. Geometry **3** (1969), 1-18.

[B–B] P.F. Baum & W. Browder, *The cohomology of quotients of classical groups*, Topology **3** (1965), 305-336.

[B–N] D. Blanc & D. Notbohm, *Mapping spaces of compact Lie groups and p–adic completion*, Proc. AMS **117** (1993), 251-258.

[Bo 1] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homoggènes de groupes de Lie compacts*, Ann. Math. **57** (1953), 115-207.

[Bo 2] A. Borel, et al., Seminar on Transformation groups, Ann. Math. Studies 46, Princeton, 1960.

[Bour] N.Bourbaki, *Groupes et Algèbres de Lie*, Hermann 1968, Chapitres 4-6.

[B–K] A. Bousfield & D. Kan, *Homotopy limits, completion and localisations*, SLNM 304, Springer Verlag, 1972.

[Ca] G. Carlsson, *Equivariant stable homotopy theory and Sullivan's conjecture*, Inventiones Math. **103** (1991), 497-525.

[C–E] Cartan & Eilenberg, *Homological algebra*, PrinctonUniversity Press 1956.

[Ch] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. math. **77** (1955), 778-782.

[Cl–Ew] A. Clark & J. Ewing, *The realzation of polynomial algebras as cohomology rings*, Pacific J. Math. **50** (1974), 425-434.

[C–W–W] M. Curtis, A. Wiederholt & B. Williams, *Normalizer of maximal tori*, Localisations in group theory and homotopy theory, Proc. of the Seattle symposium 1974, SLNM 418, Springer Verlag, 1974, pp. 31-47.

[Dr] A. Dress, *Induction and structure theorems for orthonogal representations of finite groups*, Ann. Math. **102** (1975), 291-325.

[Dw] W.G. Dwyer, *Transfer maps for fibrations*, Preprint.

[D–K] W.G. Dwyer & D. Kan, *Realizing diagrams in the homotopy category by means of diagrams of simplicial sets*, Proc. AMS **91** (1984), 456-560.

[D–K] W.G. Dwyer & D. Kan, *Centric maps and realization of disgrams in the homotopy category*, Proc. AMS **114** (1992), 575-584.

[D–M–N] W.G. Dwyer, H. Miller & J. Neisendorfer, *Fiberwise completions and unstable Adams spectral sequences*, Isreal J. Math. **66** (1989), 160-178.

[D–M–W 1] W.G. Dwyer, H.Miller, & C.W. Wilkerson, *The homotopy uniqueness of* BS^3, Algebraic Topology, Barcelona 1986,, SLNM 1298, Springer Verlag, 1987, pp. 90-105.

[D–M–W 2] W.G. Dwyer, H.Miller, & C.W. Wilkerson, *The homotopical uniqueness of classifying spaces*, Topology **31** (1992), 29-45.

[D–M] W.G. Dwyer & G. Mislin, *On the homotopy type of the components of* $map_*(BS^3, BS^3)$, Algebraic Topology, Barcelona 1986,, SLNM 1298, Springer Verlag, 1987, pp. 82-89.

[D–W 1] W.G. Dwyer & C.W. Wilkerson, *Smith theory*, Ann. Math. **127** (1988), 191-198.

[D–W 2] W.G. Dwyer and C.W. Wilkerson, *Smith theory and the functor* T, Comm. Math. Helv. **661** (1991), 1–17.

[D–W 3] W.G. Dwyer & C.W. Wilkerson, *A cohomology decomposition theorem*, Topology **31** (1992), 433-443.

[D–W 4] W.G. Dwyer & C.W. Wilkerson, *A new finite loop space at the prime two*, J. AMS **6** (1993), 37-63.

[D–W 5] W.G. Dwyer & C.W. Wilkerson, *Homotopy fixed point methods for Lie groups and finite loop spaces*, to appear in Ann. Math..

[D–W 6] W.G. Dwyer & C.W. Wilkerson, *The center of a* p*–compact group*, Preprint.

[D–W 7] W.G. Dwyer & C.W. Wilkerson, *Product splittings of* p*–compact groups*, Preprint.

[D–W 8] W.G. Dwyer & C.W. Wilkerson, *Diagrams up to cohomology*, Preprint.

[D–Z] W.G. Dwyer and A. Zabrodsky, *in "Algebraic Topology, Barcelona 1986"*, SLNM 1298, pp. 106-119.

[Fe] M. Feshbach, *The Segal conjecture for compact Lie groups*, Topology **26** (1987), 1-20.

[Fr] E. Friedlander, *Exceptional isogenies and the classifying space of simple Lie groups*, Ann. Math **101** (1975), 145-154.

[F–M 1] E. Friedlander & G. Mislin, *Locally finite approximation of Lie groups I*, Inventiones Math. **83** (1986), 425-436.

[F–M 2] E. Friedlander & G. Mislin, *Locally finite approximation of Lie groups II*, Proc. Camb. Phil. Soc. **100** (1986), 505-517.

[Hs] W.-Y. Hsiang, *Cohomology theory of transformation groups*, Springer, 1975.

[Ho] H. Hopf, *Über die Topologie der Gruppen–Mannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. Math. **13** (1940), 22-52.

[Hub 1] J. Hubbuck, *Homotopy representations of Lie groups*, new developments in Topology, London Math. Soc. Lect. Notes 11, Cambridge Univ. Press, 1974, pp. 33-41.

[Hub 2] J. Hubbuck, *Mapping degrees for classifying spaces I*, Quart. J. Math. Oxford ser (2) **25** (1974), 113-133.

[Hur] W. Hurewicz, *Beiträge zur Topologie der Deformationen IV: Asphärische Räume*, Proc. Akad. Wetensch. Amsterdam **39** (1936), 215-224.

[Is] K. Ishiguro, *Unstable Adams operations on classifying spaces*, Math. Proc. Camb. Phil. Soc. **102** (1987), 71-75.

[J–M 1] S. Jackowski & J. McClure, *Homotopy approximations for classifying spaces of compact Lie groups*, Proceedings of Arcata conference in Algebraic Topology, SLNM 1370, 1989, pp. 221-234.

[J–M 2] S. Jackowski & J. McClure, *Homotopy decomposition of classifying spaces via elememtary abelian subgroups*, Topology **31** (1992), 113-132.

[J–M–O 1] S. Jackowski, J. McClure & R. Oliver, *Homotopy classification of self maps of BG via G–actions I, II*, Ann. Math. **135** (1992), 183-226, 227-270.

[J–M–O 2] S. Jackowski, J. McClure & R. Oliver, *Maps between classifying spaces revisited*, to apear in Proceedings of the Cech centennial homotopy theory conference, AMS contemporary mathematics series.

[J–M–O 3] S. Jackowski, J. McClure & R. Oliver, *Self homotopy equivalences of classifying spaces of connected compact Lie groups*, to appear in Fundamenta Math..

[J–M–O 4] S. Jackowski, J. McClure & R. Oliver, *Homotopy theory of classifying spaces of compact Lie groups*, Algebraic Topology and its Applications, Berkeley 1989/90 (G. Carlsson et al., ed.), Math.Sc. Research Inst. Publ. 27, Springer Verlag, 1994, pp. 81-124.

[J–O] S. Jackowski & R. Oliver, *Vector bundles over classifying spaces of compact Lie groups*, Preprint.

[La 1] J. Lannes, *Sur la cohomologie modulo p des groupes abélien élémentaires*, Homotopy theory, Proceedings of the Durham symposium 1985, Cambridge University Press, 1987, pp. 97-116.

[La 2] J. Lannes, *Sur les espaces fonctionelles dont la source est la classifiant d'un p–groupe abélien eleméméntaires*, IHES **75** (1992), 135-244.

[McC–S] J. McClure & L. Smith, *On the homotopy type of $BU(2)$ at the prime 2*, Algebraic Topology, Barcelona 1990, SLNM 1509, Springer Verlag, 1992, pp. 282-284.

[McG] C. McGibbon, *Which group structures on S^3 have a maximal torus*, Geometric applications of homotopy theory I, Evanston 1977, SLNM 657, Springer Verlag, 1977, pp. 353-360.

[Mil] H.R. Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. Math.(2) **120** (1984), 39–87.

[Mis] G.Mislin, *The homotopy classification of self maps of the infinite quarternionic space*, Quart. J. Math. Oxford **38** (1987), 245-257.

[Mø1] J.M. Møller, *The normalizer of the Weyl group*, Math. Ann. **294** (1992), 59-80.

[Mø2] J.M. Møller, *Rational equivalences between classifying spaces*, to appear in Math. Z..

[Mø3] J.M. Møller, *Homotopy endomorphisms of non-connected Lie groups*, Preprint.

[Mø4] J.M. Møller, *Rational isomorphism of p–compact groups*, to appear in Topology.

[Mø5] J.M. Møller, *Extensions of p–compact groups*, to appear in Proceedings of Algebraic Topology confeerence, Barcelona 1994.

[M–N 1] J.M. Møller & D. Notbohm, *Centers and finite coverings of finite loop spaces*, to appear in J. reine u. angew. Math..

[M–N 2] J.M. Møller & D. Notbohm, *Connected finite loop spaces with maximal tori*, Math. Gott. Heft **14** (1994).

[No 1] D. Notbohm, *The fixed-point conjecture for p-toral groups*, Proceedings of the Algebraic Topology Conference Göttingen 1987, SLNM 1361, Springer Verlag, 1988, pp. 253-261.

[No 2] D. Notbohm, *Maps between classifying spaces*, Math. Z. **207** (1991), 153-168.

[No 3] D. Notbohm, *Fake Lie groups and maximal tori IV*, Math. Ann. **294** (1992), 109-116.

[No 4] D. Notbohm, *Maps between classifying spaces and applications*, J. Pure and Applied Algebra **89** (1993), 273-294.

[No 5] D. Notbohm, *Homotopy uniqueness of classifying spaces of compact connected Lie groups at primes dividing the order of the Weyl group*, Topology **33** (1994), 271-330.

[No 6] D. Notbohm, *On the functor 'Classifying space' for compact Lie groups*, to appear in J. London Math. Soc..

[No 7] D. Notbohm, *p-adic lattices of pseudo reflection groups*, Math. Gott. Heft **7** (1994).

[No 8] D. Notbohm, *Unstable splittings of classifying spaces of p–compact groups*, Math. Gott. Heft **9** (1994).

[N–S 1] D. Notbohm & L. Smith, *Fake Lie groups and maximal tori I*, Math. Ann. **288** (1990), 637-661.

[N–S 2] D. Notbohm & L. Smith, *Fake Lie groups and maximal tori II*, Math. Ann. **288** (1990), 661-673.

[N–S 3] D. Notbohm & L. Smith, *Fake Lie groups and maximal tori III*, Math. Ann. **209** (1991), 629-642.

[Ol 1] R. Oliver, *p–stubborn subgroups of the classical compact Lie groups*, J. Pure Appl. Algebra **92** (1994), 55-78.

[Ol 2] R. Oliver, *Higher limits via Steinberg representations*, Comm. in Algebra **22** (1994), 1381–1402.

[Os] A. Osse, *Sur le classifiant d'un groupe de Lie compact connexe*, C.R. Acad. Sci. Paris **315** (1992), 833-838.

[Qu 1] D. Quillen, *The spectrum of an equivariant cohomology ring I*, Ann. Math. **94** (1971), 549-572.

[Qu 2] D. Quillen, *On the cohomology and K–theory of the general linear group over a finite field*, Ann. Math. **96** (1972), 552-586.

[Re 1] D.L. Rector, *Loop structures on the homotopy type of* S^3, Symposium on Algebraic Topology, Seattle 1971, SLNM 249, Springer Verlag, 1971, pp. 99-105.

[Re 2] D.L. Rector, *Subgroups of finite dimensional topological groups*, J. pure Appl. Algebra **4** (1971), 253-273.

[Re 3] D.L. Rector, *Noetherian cohomology rings and finite loop spaces with torsion*, J. Pure Appl. Algebra **32** (1984), 191–227.

[Se] G. Segal, *Classifying spaces and spectral sequences*, Publ. Math. I.H.E.S. **34** (1968), 105-112.

[St] N.E. Steenrod, *Polynomial algebras over the algebra of cohomology operations*, Proceedings, Neuchâtel 1970, SLNM 196, Springer Verlag, 1971.

[Su] D. Sullivan, *Geometric Topology, Part I: Localisations, periodicity and Galois theory*, Mimeographed notes, MIT (1970).

[Ve] B.B. Venkov, *Cohomology algebras for some classifying spaces*, Dokl. Akad. Nauk. SSSR **127** (1959), 943-944.

[Wi 1] C.W. Wilkerson, *Self maps of classifying spaces*, Localisations in group theory and homotopy theory, SLNM 418, Springer Verlag, 1974 , pp. 150-157.

[Wi 2] C. Wilkerson, *Rational maximal tori*, J. Pure Appl. Algebra **4** (1974), 261–272.

[Wo 1] Z. Wojtkowiak, *On maps from holim F to Z*, Algebraic Topology, Barcelona 1986, SLNM 1298, Springer Verlag, 1987, pp. 227-236.

[Wo 2] Z. Wojtkowiak, *Maps between p–completions of the Clark–Ewing spaces* $X(W,p,n)$, S.M.F Astérisque **191** (1990), 269-284.

[Wo 3] Z. Wojtkowiak, *Maps between p–completed classifying spaces III* , Preprint.

[Za 1] A. Zabrodsky, *On the realization of invariant subgroups of* $\pi_*(X)$, Trans. AMS **285** (1984), 467-496.

[Za 2] A. Zabrodsky, *Maps between classifying spaces*, Algebraic Topology and Algebraic K-theory, Ann. of Math. Studies 113, 1987, pp. 228-246.

[Za 3] A. Zabrodsky, *On phantom maps and a theorem of H. Miller*, Israel J. Math. **58** (1987), 129–143.

[Za 4] A. Zabrodsky, *On the space of functions between classifying spaces*, Israel J. Math. **76** (1991), 1-26.

Mathematsisches Institut, Bunssenstr. 3-5, 37073 Göttingen, Germany.
e-mail : notbohm at cfgauss.uni-math.gwdg.de