A VECTOR BUNDLE OVER DAVIS-JANUSZKIEWICZ SPACES AND COLORINGS OF SIMPLICIAL COMPLEXES

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ABSTRACT. We study a particular complex vector bundle over Davis-Januszkiewicz spaces. We show that the isomorphism class of this vector bundle is complety determined by its Chern classes and the isomorphism class of its realification by its Pontrjagin and Euler classes. As shown by Davis and Januczkiewicz, this vector bundle is closely related to the equivariant tangent bundle of toric manifolds respectively of moment angle complexes. Moreover, we show that splitting properties of the vector bundle are reflected by colorings of simplicial complexes and vice versa.

1. INTRODUCTION

Given a simplicial complex K, Davis and Januszkiewicz constructed a family of spaces, all of which are homotopy equivalent, and whose integral cohomology is isomorphic to the associated Stanley-Reisner algebra $\mathbb{Z}[K]$ [DJ]. We denote a generic model for this homotopy type by $DJ(K)$. In the above mentioned influential paper, Davis and Januszkiewicz also constructed a particular complex vector bundle λ over $DJ(K)$ whose Chern classes are given by the elementary symmetric polynomials in the generators of $\mathbb{Z}[K]$. This vector bundle is of particular interest. For example, if K is the dual of a simple polytope P, the realification $\lambda_{\mathbb{R}}$ of λ is stably isomorphic to the bundle given by applying the Borel construction to the tangent bundle of the associated moment angle complex Z_K [DJ]. And if M^{2n} is a toric manifold over P, then again the Borel construction applied to the tangent bundle of M^{2n} produces a vector bundle stably isomorphic to $\lambda_{\mathbb{R}}$.

We will split of a large trivial vector bundle of λ. We are interested in two aspects of the remaining vector bundle ξ. We will show that the Chern classes $c_i(\xi)$ of ξ determine ξ up to isomorphism, and the Pontrjagin classes $p_j(\xi)$ together with Euler class $e(\xi)$ do the same for $\xi_{\mathbb{R}}$. We will also show that stable splittings of ξ into a direct sum of linear complex bundles respectively of ξ_R into a direct sum of 2-dimensional real bundles produce colorings of K and vice versa.

To make our statements more precise we have to fix notation and recall some basic constructions. Let $[m] := \{1, ..., m\}$ be the set of the first m

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natural numbers. A finite abstract simplicial complex K on $[m]$ is given by a set of faces $\alpha \subseteq [m]$ which is closed under the formation of subsets. We consider the empty set $\emptyset$ as a face of K. The dimension $\dim(\alpha)$ of a face α is given in terms of its cardinality by $|\alpha|-1$, and the dimension $\dim(K)$ of K is given by the maximum of the dimensions of its faces. The most basic example is given by the full simplex $\Delta(m)$ which consists of all possible subsets of $[m]$. In particular, $\Delta(m)$ contains K as a subcomplex.

For a commutative ring R with unit we denote by $R[m] := R[v_1, ..., v_m]$ the graded polynomial algebra generated by the algbraicaly independent elements $v_1, ..., v_m$ of degree 2, one for each vertex of K. For each subset $\alpha \subset [m]$ we denote by $v_\alpha := \prod_{j\in\alpha} v_j$ the square free monomial whose factors are in 1 to 1 relation with vertices contained in α. The graded Stanley-Reisner algebra $R[K]$ associated with K is defined as the quotient $R[K] := R[m]/I_K$, where $I_K \subset R[V]$ is the ideal generated by all elements v_μ such that $\mu \subset [m]$ is not a face of K.

For a complex vector bundle η we denote by $c(\eta)$ the total Chern class. And for a real vector bundle ρ we denote by $p(\rho)$ the total Pontrjagin class and, if ρ is oriented, by $e(\rho)$ the Euler class.

Theorem 1.1. *Let K be a finite simplicial complex of dimension $n-1$.*
(i) There exists an n-dimensional complex vector bundle ξ over $DJ(K)$ such that $c(\xi) = \prod_{i=1}^m (1+v_i) \in \mathbb{Z}[K]$ and $p(\xi_\mathbb{R}) = \prod_{i=1}^m (1-v_i^2) \in \mathbb{Z}[K]$.
(ii) If $\eta \downarrow DJ(K)$ is another n-dimensional complex vector bundle over $DJ(K)$ such that $c(\eta) = c(\xi)$, then ξ and η are isomorphic.
(iii) If $\rho \downarrow DJ(K)$ is another oriented real vector bundle over $DJ(K)$ such that $p(\rho) = p(\xi_\mathbb{R})$ then ρ and $\xi_\mathbb{R}$ are isomorphic as (unoriented) real vector bundles. If in addition $e(\rho) = e(\xi_\mathbb{R})$, then ρ and $\xi_\mathbb{R}$ are isomorphic as oriented real vector bundles.

We will relate splitting properties of the vector bundle $\xi \downarrow DJ(K)$ to colorings of K. A regular r-paint coloring, an r-coloring for short, of a simplicial complex K is a non degenerate simplicial map $g : K \longrightarrow \Delta(r)$, i.e. the restriction of g on each face is an injection. If $\dim(K) = n-1$, then K only allows r-colorings for $r \geq n$. The inclusion $K \subset \Delta(m)$ always provides an m-coloring. Confusing notation we will denote by $\mathbb{C}$ and $\mathbb{R}$ a 1-dimensional trivial (G-equivariant) complex or real vector bundles over a (G-)space X.

Theorem 1.2. *Let K be a finite simplicial complex of dimension $n-1$ and let $r \geq n$. Then the following statements are equivalent:*
(i) There exists an r-coloring for K.
(ii) The complex vector bundle $\xi \oplus \mathbb{C}^{r-n} \downarrow DJ(K)$ splits into a direct sum of complex line bundles.
(iii) The real vector bundle $\xi_\mathbb{R} \oplus \mathbb{R}^{2(r-n)}$ splits into a direct sum of 2-dimensional real vector bundles.

Remark 1.3. Let M^{2n} be a quasi toric manifold over the simple polytope P. That is that M^{2n} carries a T^n-action, which is locally standard and that $M^{2n}/T^n = P$ is a simple polytope. The Borel construction produces a space $(M^{2n})_{hT^n} := ET^n \times_{T^n} M^{2n}$, which is homotopy equivalent to $DJ(K)$, where K is the dual simplicial complex of P. For details see [DJ] or [BP]. Davis and Januczkiewicz showed that the Borel construction applied to the tangent bundle τ of M^{2n} produces an oriented vector bundle $\tau_{hT^n} \downarrow DJ(K)$ with the same Pontrjagin classes as ξ. Hence, by Theorem 1.1 both are isomorphic. Actually, in this case Davis and Januczkiewicz already constructed the bundle ξ and showed that $\xi_{\mathbb{R}}$ and τ_{hT^n} are isomorphic [DJ]. For details of the construction of τ_{hT^n} see Section 3.

In particular, if $\tau \oplus \mathbb{R}^{2(r-n)}$ splits equivariantly into a direct sum of 2-dimensional T^n-equivariant real vector bundle, then there exists an r-coloring for K. The opposite conclusion does not hold in general, since, for a T^n-equivariant vector bundle $\eta \downarrow M^{2n}$, a splitting of η_{hT^n} may not imply an equivariant splitting of η.

If the moment angle complex Z_K is a manifold, a similar considerations applies to the tangent bundle of Z_K.

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The paper is organized as follow. In the next section we describe different models for $DJ(K)$ needed for our purposes. The geometric construction of the vector bundle ξ with the properties stated in Theorem 1.1(i) is contained in Section 3. Using Sullivan's arithmetique we reduce the global uniqueness problem to the analogue p-adic question in Section 4. The p-adic uniqueness problem, discussed in Section 5, involves the calculation of some higher derived inverse limits. These calculation are worked out in Section 6 and finish the proof of Theorem 1.1. in the final section we put splitting properties of ξ and colorings of K in relation and prove Theorem 1.2.

If not otherwise specified, K will always denote an abstract finite simplicial complex of dimension $n-1$ with m vertices.

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2. Models for $DJ(K)$

For the proof of our Theorems we will need different models for the space $DJ(K)$, which we will describe in this section.

Given a pair (X, Y) of pointed topological space we can define covariant functors

$$X^K,\ (X,Y)^K,\ \mathrm{cst}_X : \mathrm{CAT}(K) \longrightarrow \mathrm{TOP}.$$

The functor X^K assigns to each face α the cartesian product X^α and to each morphism $i_{\alpha,\beta}$ the inclusion $X^\alpha \subset X^\beta$ where missing coordinates are set to $*$. And $(X,Y)^K$ assigns to α the product $X^\alpha \times Y^{[m]\setminus\alpha}$ and to $i_{\alpha,\beta}$ the coordinate wise inclusion $X^\alpha \times Y^{[m]\setminus\alpha} \subset X^\beta \times Y^{[m]\setminus\beta}$. The constant functor cst_X assigns

X to each face and the identity id_X to each morphism. The inclusions $X^\alpha \subset X^{[m]} = X^m$ establish a natural transformation $X^K \longrightarrow \mathrm{cst}_{X^m}$.

We are interested in two particular cases, namely the functor X^K for the classifying space $BT = \mathbb{C}P(\infty)$ of the 1-dimensional circle T and the functor $(X,Y)^K$ for the pair (D^2, S^1). The colimit

$$Z_K := \mathrm{colim}_{\mathrm{CAT}(K)}\,(D^2, S^1)^K$$

is called the moment angle complex associated to K. We have inclusions $Z_K \subset (D^2)^m \subset \mathbb{C}^m$ and the standard T^m-action on $\mathbb{C}^m$ restricts to Z_K. The Borel construction produces a fibration

$$q_K : (Z_K)_{hT^m} := ET \times_{T^m} Z_K \longrightarrow BT^m$$

with fiber Z_K. Moreover, $DJ(K) := Z_K)_{hT^m}$ is a realization of the Stanley-Reisner algebra $\mathbb{Z}[K]$ and such that the map $H^*(q_K;\mathbb{Z})$ can be identified with the map $\mathbb{Z}[m] \longrightarrow \mathbb{Z}[K]$ [DJ]. We will use this model for the geometric construction of our vector bundle ξ.

In [BP], Buchstaber and Panov showed that $c(K) := \mathrm{colim}_{\mathrm{CAT}(K)}\ BT^K$ is homotopy equivalent to $DJ(K)$ and that the map

$$c(K) \longrightarrow \mathrm{colim}_{\mathrm{CAT}(K)}\ \mathrm{cst}_{BT^m} = BT^m$$

is homotopic to q_K. We wish to study homotopy theoretic properties of $DJ(K) \simeq c(K)$. Colimits behave poorly from a homotopy theoretic point of view, but the left derived functor, known as homotopy colimit, provides the appropriate tool for such questions. Following [V], the homotopy colimit $hc(K) := \mathrm{hocolim}_{\mathrm{CAT}(K)}\ BT^K$ may be described as the two sided bar construction $B(*, \mathrm{CAT}(K), BT^K)$ in TOP. For the functor BT^K, the projection

$$\mathrm{hocolim}_{\mathrm{CAT}(K)}\ BT^K \longrightarrow \mathrm{colim}_{\mathrm{CAT}(K)}\ BT^K$$

is a homotopy equivalence [NR1]. In particular, $hc(K) \simeq c(K) \simeq DJ(K)$. We will use the model $hc(K)$ to prove the uniqueness properties of the vector bundle ξ and to provide a homotopy theoretic construction of ξ.

3. A GEOMETRIC CONSTRUCTION OF THE VECTOR BUNDLE ξ

For particular cases of simplicial complexes of dimension $n-1$, Davis and Januszkiewicz constructed an n-dimensional complex vector bundle over $DJ(K)$ by geometric means. The total Chern class of this bundle is given by $\prod_i (1+v_i) \in \mathbb{Z}[K]$. We will adjust their construction to our needs and slightly generalize it.

Let G be a compact Lie group and X a G-space. We will denote the Borel construction $EG \times_G X$ by X_{hG}. If $\eta \downarrow X$ is an n-dimensional G-vector bundle over X with total space $E(\eta)$, the Borel construction establishes a fibre bundle $E(\eta)_{hG} \longrightarrow X_{hG}$. In fact, this is an n-dimensional vector bundle over X_{hG} [S], denoted by η_{hG}.

Now we can start with our construction. Let L_i denote the complex 1-dimensional T^m-representation given by the T^m-action on $\mathbb{C}$ via the i-th coordinate. We set $L := \bigoplus_{i=1}^m L_i \cong \mathbb{C}^m$. The diagonal action of T^m on the products $L_i \times Z_K$ and $L \times Z_K$ and the projections onto the second factor establish complex T^m-equivariant vector bundles $\lambda_i' \downarrow Z_K$ and $\lambda' \downarrow Z_K$ over Z_K. By construction, we have $\lambda' \cong \bigoplus_i \lambda_i'$. We define $\lambda_i := (\lambda_i')_{hT^m}$ and $\lambda := \lambda'_{hT^m}$. It is straight forward to see that the total Chern class $c(\lambda_i)$ is given by $1 + v_i \in \mathbb{Z}[K]$, that $c(\lambda) = \prod_i (1 + v_i) \in \mathbb{Z}[K]$, that $p(\lambda_i) = 1 - v_i^2 \in \mathbb{Z}[K]$ and that $p(\lambda) = \prod_i (1 - v_i^2) \in \mathbb{Z}[K]$. (see [DJ]).

Let $A \in \mathbb{C}^{m\times(m-n)}$ be a $m \times (m-n)$-matrix such that each square submatrix given by an $m-n$ rows is invertible, e.g. we can take $A = (s^r)$, $1 \le r \le m,\ 1 \le s \le m-n$. We define a map

$$f_A : \mathbb{C}^{m-n} \times Z_K \longrightarrow \mathbb{C}^m \times Z_K$$

by $f_A(x,z) := (y,z)$ where $y := Ax\cdot z$ is given by coordinate wise multiplication of Ax and z, i.e. the i-th coordinate y_i of y is given by $y_i := (Ax)_i z_i$. Here, we use that $Z_K \subset \mathbb{C}^m$. If T^m acts on the source only via the second coordinate, then $E(m-n) := \mathbb{C}^{m-n} \times Z_K$ is the total space of the $m-n$-dimensional trivial T^m-bundle over $\mathbb{C}^{m-n} \downarrow Z_K$. The target is the total space $E(\lambda')$ of the T^m-bundle λ'.

The following proposition contains Part (i) of Theorem 1.1. We will give a different proof for it in the following sections.

Proposition 3.1.
(i) The map $f : E(m-n) \longrightarrow E(\lambda')$ is T^m-equivariant and a bundle monomorphism. Moreover, $\lambda' \cong \mathbb{C}^{m-n} \oplus \eta$ as T^m-equivariant vector bundles for an appropriate T^m-equivariant n-dimensional complex vector bundle $\eta \downarrow Z_K$.
(ii) $\lambda = \lambda'_{hT^m} \cong \mathbb{C}^{m-n} \oplus \eta_{hT^m}$.
(iii) The total Chern class of $\xi := \eta_{hT^m}$ is given by $c(\xi) = \prod_i (1+v_i) \in \mathbb{Z}[K]$.
(iv) The total Pontrjagin class of $\xi := \eta_{hT^m}$ is given by $p(\xi) = \prod_i (1 - v_i^2) \in \mathbb{Z}[K]$.

Proof. For $t \in T^m$, we have $f_A(t(x,z) = f(x, t(z)) = (Ax \cdot t(z), t(z)) = t(Ax \cdot z, z) = t f_A(x,z)$. This shows that f_A is T^m-equivariant. The map f_A is a bundle monomorphism, since at least $m-n$ coordinates of z are unequal to 0, and since any set of $m-n$ rows of A make up an invertible square matrix. The quotient $\eta := \lambda/(\mathbb{C}^{m-n})$ is again a T^m-vector bundle over Z_K. Since every short exact sequence of T^m-equivariant bundles of the compact space Z_K splits [S], we get $\lambda' \cong \eta \oplus \mathbb{C}^{m-n}$. This proves the first part. The other three parts are direct consequences of Part (i). □

4. Uniqueness properties of ξ

In this section we want to prove the second and third part of Theorem 1.1. As a side effect we will also provide a homotopy theoretic proof of the first part. We will reformulate Theorem 1.1 in terms of homotopy theory and describe a r-dimensional vector bundle $\nu \downarrow X$ over a topological space X by its classifying map $\nu : X \longrightarrow BU(r)$.

We can identify the integral cohomology of the classifying spaces BT^m and $BU(m)$ of with $\mathbb{Z}[m]$ and $\mathbb{Z}[c_1, ..., c_m]$, where c_i denotes the i-th universal Chern class. Passing to classifying spaces and cohomology, the standard inclusion $j : T^m = T_{U(m)} \subset U(m)$ of the maximal torus induces the map $\mathbb{Z}[c_1, ..., c_m] \cong \mathbb{Z}[m]^{\Sigma_m} \subset \mathbb{Z}[m]$, where Σ_m denotes the symmetric group on m letters respectively the Weyl group of $U(m)$. If K is a finite simplicial complex of dimension $n-1$ and $n \leq r \leq m$, the composition $\mathbb{Z}[c_1, ..., c_m] \longrightarrow \mathbb{Z}[m] \longrightarrow \mathbb{Z}[K]$ maps the Chern class c_j to zero for $j \geq r+1$. Hence this composition factors uniquely through $\mathbb{Z}[c_1, ..., c_r]$ and establishes a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[c_1, ..., c_m] & \longrightarrow & \mathbb{Z}[m] \\ \downarrow & & \downarrow \\ \mathbb{Z}[c_1, ..., c_r] & \xrightarrow{g_r} & \mathbb{Z}[K] \end{array}$$

This diagram can partly be realized by spaces and maps, namely by

$$\begin{array}{ccc} BU(m) & \longleftarrow & BT^m \\ \uparrow & & \uparrow \\ BU(r) & & c(K) \end{array}$$

We want to construct a map $c(K) \longrightarrow BU(r)$ making the above diagram commutative up to homotopy. We can think of such a map as the classifying map of an n-dimensional complex vector bundle over $DJ(K)$ which has the desired Chern classes. In fact, we want to show that such a map is unique up to homotopy.

Let $U(r) \longrightarrow SO(2r)$ and $SO(2r) \longrightarrow O(2r)$ denote the standard inclusions. Composition with $BU(r) \longrightarrow BSO(2r)$ reflects the realification of a complex vector bundle $\eta : X \longrightarrow BU(r)$ and will be denoted by $\eta_{\mathbb{R}}$. Composition with $BSO(2r) \longrightarrow BO(2r)$ means we forget the orientation of a vector bundle. For real oriented vector bundle $\rho : X \longrightarrow BSO(2r)$, this will be denoted by $\overline{\rho}$ The composition

$$T^r \longrightarrow U(r) \longrightarrow SO(2r) \longrightarrow O(2r)$$

describes at each stage a maximal torus of the compact Lie group. Passing to classifying spaces and and rational cohomology,

$$H^*(BO(2r); \mathbb{Q}) \cong \mathbb{Q}[m]^{W_{O(2r)}} \cong \mathbb{Q}[p_1, ..., p_r]$$

is a polynomial algebra generated by the universal Pontrjagin classes p_i and

$$H^*(BSO(2r); \mathbb{Q}) \cong \mathbb{Q}[m]^{W_{SO(2r)}} \cong \mathbb{Q}[p_1, ..., p_{r-1}, e]$$

is a polynomial algebra generated by the first $r-1$ Pontrjagin classes and the universal Euler class e.

The following theorem is a homotopy theoretic formulation of Theorem 1.1.

Theorem 4.1.
(i) For $n \leq r$, there exist a map $\xi_r : DJ(K) \longrightarrow BU(r)$, such that $H^*(\xi_r;\mathbb{Z}) = g_r$.
(ii) A map $\eta : DJ(K) \longrightarrow BU(r)$ is homotopic to ξ_r if and only if $H^*(\eta;\mathbb{Q}) = H^*(\xi_r;\mathbb{Q})$.
(iii) For a map $\rho : DJ(K) \longrightarrow BSO(2r)$ the composition $\overline{\rho}$ is homotopic to $(\overline{\xi_r})_{\mathbb{R}}$ if and only if $H^*((\overline{\xi_r})_{\mathbb{R}};\mathbb{Q}) = H^*(\overline{\rho};\mathbb{Q})$ And ρ is homotopic to $(\xi_r)_{\mathbb{R}}$ if and only $H^*((\xi_r)_{\mathbb{R}};\mathbb{Q}) = H^*(\rho;\mathbb{Q})$.

Remark 4.2. Let $\rho : DJ(K) \longrightarrow BSO(2r)$ be a real oriented vector bundle such that $p(\rho) = p((\xi_r)_{\mathbb{R}})$. Since $e^2 = (-1)^n p_r \in H^*(BSO(2r);\mathbb{Q})$, the Euler classes $e(\rho)$ and $e((\xi_r)_{\mathbb{R}})$ may only differ by a sign. Hence changing the orientation of ρ we can always achieve that $e(\rho) = e((\xi_r)_{\mathbb{R}}$. Therefore we only have to show that the Pontrjagin classes together with the Euler class characterize $(\xi_r)_{\mathbb{R}}$ up to isomorphisms of oriented real vector bundles.

The proof is based on arithmetique square arguments and similar statements over the rationals and over p-adic integers, each prime at a time. For a topological space X we denote by X_0 the rationalization, by $X_p^\wedge$ the p-adic completion in the sense of Bousfield and Kan and by $X_{\mathbb{A}_f}$ the finite adele type of X. If X is 'nice' (and all spaces under consideration are 'nice'), these spaces fit together to Sullivan's arithmetique square

$$\begin{array}{ccc} X & \longrightarrow & X^\wedge := \prod_p X_p^\wedge \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & X_{\mathbb{A}_f} \end{array}$$

In the following the group G_r denotes either $U(r)$ or $SO(2r)$. And confusing notation we will denote the composition $(\xi_r)_{\mathbb{R}}$ by ξ_r as well.

Theorem 4.3. *Under the assumption of Theorem 4.1 the following holds:*
(i) There exists a map $(\xi_r)_0 : DJ(K) \longrightarrow BU(r)_0$, such that $H^*((\xi_r)_0;\mathbb{Q}) = g_r \otimes \mathbb{Q}$.
(ii) A map $\rho : DJ(K) \longrightarrow BG(r)_0$ is homotopic to $(\xi_r)_0$ if and only if $H^*(\rho;\mathbb{Q}) = H^*(\xi_r;\mathbb{Q})$.

Theorem 4.4. *Let p be a prime. Under the assumptions of Theorem 4.1 the following holds:*
(i) There exists a map $(\xi_r)_p^\wedge : DJ(K) \longrightarrow BU(r)_p^\wedge$, such that $H^*((\xi_r)_p^\wedge;\mathbb{Z}_p^\wedge) = g_r \otimes \mathbb{Z}_p^\wedge$.
(ii) A map $\rho : DJ(K) \longrightarrow BG(r)_p^\wedge$ is homotopic to $(\xi_r)_p^\wedge$ if and only if $H^*(\rho;\mathbb{Z}_p^\wedge) = H^*(\xi_r;\mathbb{Z}_p^\wedge)$.

We will also need

Theorem 4.5. *The map*

$$[c(K), BG(r)] \longrightarrow [DJ(K), BG(r)^\wedge] = \prod_p [DJ(K), BG(r)_p^\wedge]$$

is a monomorphism.

Proof of Theorem 4.1: Rationally, $BG(r)_0$ is a product of rational Eilenberg-MacLane spaces of even degree, one factor for each generator of $H^*(BG(r);\mathbb{Q})$. Therefore, the same holds for the finite adele type $BG(r)_{\mathbb{A}_f}$ and, up to homotopy, maps into $BG(r)_{\mathbb{A}_f}$ are determined by cohomological information, Since $(\xi_r)_0$ and $(\xi_r)_p^\wedge$ realize $g_r \otimes \mathbb{Q}$ respectively $g_r \otimes \mathbb{Z}_p^\wedge$, the compositions

$$c(K) \xrightarrow{(\xi_r)_0} BU(n)_0 \longrightarrow BU(n)_{\mathbb{A}_f}$$

and

$$c(K) \xrightarrow{(\xi_r)_p^\wedge} \prod_p BU(n)_p^\wedge = BU(n)^\wedge \longrightarrow BU(n)_{\mathbb{A}_f}$$

are homotopic. Using the arithmetique square for $BG(r)$ we can construct map $\xi_r : DJ(K) \longrightarrow BG(r)$ with the desired cohomological property.

Since $H^*(DJ(K);\mathbb{Z})$ is torsion free, the map $H^*(\rho;\mathbb{Q})$ determines $H^*(\rho;\mathbb{Z})$ as well as $H^*(\rho;\mathbb{Z}_p^\wedge)$. The homotopical uniqueness of ξ_r follows from Theorem 4.4 and Theorem 4.5. □

Proof of Theorem 4.3: This follows from the fact that $BG(r)_0$ is a product of rational Eilenberg-MacLane spaces. □

Proof of Theorem 4.5: The homotopy fiber F of $BG(r) \longrightarrow BG(r)^\wedge$ is equivalent to the homotopy fiber of $BG(r)_0 \longrightarrow BG(r)_{\mathbb{A}_f}$. Since $BG(r)_0$ is a product of rational Eilenberg-MacLane spaces of even degree, $\pi_i(F) = 0$ for i even. The obstruction groups for lifting homotopies between maps $DJ(K) \longrightarrow BG(r)_p^\wedge$ to $BG(r)$ are given by $H^*(DJ(K);\pi_*(F))$. All these obstruction groups vanish, since $H^*(DJ(K);\mathbb{Z})$ is concentrated in even degrees. □

The proof of Theorem 4.4 is contained in the next section.

5. P-ADIC HOMOTOPY UNIQUENESS

In this section we will work with the model $hc(K) = \text{hocolim}_{\text{CAT}(K)} BT^K$ for $DJ(K)$. All homotopy colimits are taken over $\text{CAT}(K)$ and all higher derived limits over the opposite category $\text{CAT}(K)^{op}$. For simplification we will drop these categories in the notations of limits. We will also skip the notation of completion for maps. Again, $G(r)$ will denote the either $U(r)$ or $SO(2r)$ and $T^r \subset G(r)$ the standard maximal torus.

For each face $\alpha \in K$ we denote by $\mathbb{Z}[\alpha]$ the polynomial algebra with one generators in degree 2 for each element of α. The algebra map

$$\mathbb{Z}[c_1, ..., c_r] \xrightarrow{g_r} \mathbb{Z}[K] \longrightarrow \mathbb{Z}[\alpha]$$

maps c_i onto the i-the elementary symmetric polynomial in $\mathbb{Z}[\alpha]$. It can be realized by a composition

$$\xi_r^\alpha : BT^\alpha \longrightarrow BT^r \longrightarrow BU(r),$$

where the second map is induced by the maximal torus inclusion $T^r \subset U(r)$ and the first map by a coordinate wise inclusion $T^\alpha \longrightarrow T^r$. Different coordinate wise inclusion into T^r only differ by a permutation and hence by an element of the Weyl group of $U(r)$. They are therefore conjugate in $U(r)$ and induce homotopic maps $BT^\alpha \longrightarrow BU(r)$ between the classifying spaces. Moreover, since

$$[BT^\alpha, BU(n)^\wedge_p] \longrightarrow Hom(H^*(BU(n); \mathbb{Z}^\wedge_p), H^*(BT^\sigma; \mathbb{Z}^\wedge_p))$$

is injective [N], the algebra determines a unique homotopy type of a map $\xi_r^\alpha : BT^\alpha \longrightarrow BU(r)^\wedge_p$. In particular, for an inclusion $\alpha \subset \beta$ the triangle

$$\begin{array}{ccc} BT^\alpha & \longrightarrow & BT^\beta \\ & {}_{\xi_r^\alpha}\searrow \quad \swarrow_{\xi_r^\beta} & \\ & BU(r)^\wedge_p & \end{array}$$

commutes up to homotopy. This defines a map $\xi_r^{(1)} : hc(K)^{(1)} \longrightarrow BU(r)^\wedge_p$, unique up to homotopy, on the 1-skeleton of the homotopy colimit $hc(K)$ into $BU(n)^\wedge_p$. The Bousfield-Kan spectral sequence for homotopy inverse limits [BK], together with work by Wojtkowiak [W] clarifying the situation for the fundamental group, provides an obstruction theory for extending this map to $hc(K)$. The obstruction groups are given by the higher derived limits

$$\lim{}^{i+1} \Pi_i^U,$$

where $\Pi_i^U : \text{CAT}(K)^{op} \longrightarrow \text{AB}$ is the functor defined by

$$\Pi_i^U(\alpha) := \pi_i(map(BT^\alpha, BU(n)^\wedge_p)_{\xi_r^\alpha}$$

on faces α and the induced homomorphisms on inclusions. For a definition and properties of higher derived limits see [BK] or [O].

The obstruction theory may also decide the question of uniqueness. Let $\xi_r : DJ(K) \longrightarrow BG(r)$ denote the map under consideration in both cases and let Π_i^G denote the functor given by $\Pi_i^G(\alpha) := map\,(BT^\alpha, BG(r)^\wedge_p)_{\xi_r^\alpha}$. The image of the restriction map

$$R : [hc(k), BG(r)^\wedge_p] \longrightarrow \lim{}^0\, [BT^K, BG(r)^\wedge_p]$$

may be identified with homotopy classes of maps $hc(K)^{(1)} \longrightarrow BG(r)^\wedge_p$. If $\lim^i \Pi_i^G$ vanishes for all $i \geq 1$, then $R^{-1}(\xi_r^{(1)})$ consist of at most one element.

By the above consideration, Theorem 4.4 is a consequence of

Proposition 5.1. *For all $j \geq i \geq 1$, we have*

$$\lim{}^j \Pi_i^G = 0$$

Remark 5.2. Let $\rho : DJ(K) \longrightarrow BG(r)$ be a map such that for each face α of K the restriction $\rho|_{BT^\alpha}$ of ρ to BT^α is homotopic to the restriction $\xi_r|_{BT^\alpha}$. Then, the proof of Theorem 4.4 shows that the two maps $\rho^\wedge_p$, ${\xi_r}^\wedge_p$:

$DJ(K) \longrightarrow BG(r)^\wedge_p$ are homotopic for all primes and Theorem 4.5 implies that ρ and ξ_r are homotopic.

We will prove Proposition 5.1 in the next section, where we will discuss some general procedures to calculate higher derived limits of functors defined on $\text{CAT}(K)^{op}$. We will finish this section by collecting the data necessary for the proof of the above proposition.

The involved mapping spaces can be calculated. The map $(\xi_r)_\alpha$ is induced by a coordinate wise inclusion $\iota^\alpha : T^\alpha \longrightarrow T_{G(r)} \longrightarrow G(r)$ into the maximal torus of $G(r)$. The centralizer $C_{G(r)}(T^\alpha) := C_{G(r)}(\iota^\alpha)$ of the image of ι^α is given by $T^\alpha \times G(n-|\alpha|)$. By [N], there exists a mod-p equivalence

$$BT^\alpha \times BG(r-|\alpha|) = BC_{G(r)}(j_\alpha) \longrightarrow map(BT^\alpha, BG(r)^\wedge_p)_{(\xi_r)_\alpha}.$$

Moreover, up to homotopy the above mod-p equivalence is natural with respect to the morphisms $\beta \supset$
$alpha$ in $\text{CAT}(K)^{op}$. Such an inclusion induces the composition

$$BT^\beta \times BG(r-|\beta|) \longrightarrow BT^\alpha \times BT^{\alpha\backslash\beta} \times BG(r-|\beta|) \longrightarrow BT^\alpha \times BG(r-|alpha|).$$

between the classifying spaces of the centralizers. After p-adic completion this map is equivalent to the induced map between the associated mapping spaces and, passing to homotopy groups, describes the map

$$\Pi_i^G(\alpha) \longrightarrow \Pi_i^G(\beta).$$

It will be convenient to define functors

$$\Psi_2, \hat{\Pi}_2^G : \text{CAT}(K)^{op} \longrightarrow \text{AB}$$

by $\Psi_2(\alpha) := \pi_2((BT^\alpha)^\wedge_p)$ and $\hat{\Pi}_2^G(\alpha) := \pi_2(BG(r-|\alpha|)^\wedge_p)$.

Lemma 5.3.
(i) For $i=2$, we have an exact sequence

$$0 \longrightarrow \hat{\Pi}_2^G \longrightarrow \Pi_2^G \longrightarrow \Psi_2 \longrightarrow 0$$

of functors. And $\Psi_2 \cong H^2(BT^K; \mathbb{Z}_p^\wedge)$.
(ii) $\Pi_{2j+1}^U(\alpha) = 0$ for all $j \leq r - |\alpha| - 1$.
(iii) If $j \geq 2$, $\alpha \subset \beta$ and $|\beta| \leq n - j$, then $\Pi_{2j}^U(\beta) \cong \Pi_{2j}^U(\alpha)$. If $j = 1$, the same formula holds for $\hat{\Pi}_2^G$.
(iv) $\Pi_1^{SO}(\alpha) = 0$ for all α.
(v)If p is odd, then $\hat{\Pi}_2^{SO}(\alpha) = 0$ for $|\alpha| \neq r-1$.
(vi) If $t \geq 3$ and $\alpha \subset \beta$, then $\Pi_t^{SO}(\beta) \longrightarrow \Pi_t^{SO}(\alpha)$ is an isomorphism for $|\beta| \leq n - t/2 - 1/2$. If $p = 2$, the same formula holds for $\hat{\Pi}_2^{SO}$.

Proof. The first half of Part (i) follows from the above considerations, the second half is obvious.

Part (ii) and Part (iii) follow from the fact that $\pi_{2s+1}(BU(t)) = 0$ for $0 \leq s \leq t-1$ and $\pi_{2s}(BU(t)) \cong \pi_{2s}(BU(t+1))$ for $1 \leq s \leq t$.

For every connected compact Lie group H, the classifying space BH is simply connected. This proves Part (iv).

The fifth part follows from the fact that $\pi_2(BSO(s)) \cong \mathbb{Z}/2$ for $s \geq 3$.

Finally, $\pi_t(BSO(k)) \longrightarrow \pi_t(BSO(k+1))$ is an isomorphism for $t \leq k-1$. Since $\Pi_t^{SO}(\beta) = \pi_t(BSO(2(n-|\beta|)_p^\wedge)$, this implies the last claim. □

6. HIGHER LIMITS

We will continue to drop $\text{CAT}(K)^{op}$ in the notation of limits. Given a functor $\Phi : \text{CAT}(K)^{op} \longrightarrow \text{AB}$ we define for $0 \leq s \leq n$ functors $\Phi_{\leq s}, \Phi_s : \text{CAT}(K)^{op} \longrightarrow \text{AB}$ by

$$\Phi_{\leq s}(\alpha) := \begin{cases} \Phi(\alpha) & \text{for } |\alpha| \leq s \\ 0 & \text{for } |\alpha| > s \end{cases} \qquad \Phi_s(\alpha) := \begin{cases} \Phi(\alpha) & \text{for } |\alpha| = s \\ 0 & \text{for } |\alpha| \neq s \end{cases}$$

Since for $|\alpha| \leq |\beta|$ there is no arrow $\alpha \to \beta$ in $\text{CAT}(K)^{op}$, both functors are well defined. Moreover, we have $\Phi_{\leq n} = \Phi$. There exist exact sequences of functors

$$1 \longrightarrow \Phi_{\leq s-1} \longrightarrow \Phi_{\leq s} \longrightarrow \Phi_s \longrightarrow 1$$

which induce long exact sequences

$$\ldots \to \lim{}^{i-1} \Phi_s \to \lim{}^i \Phi_{\leq s-1} \to \lim{}^i \Phi_{\leq s} \to \lim{}^i \Phi_s \longrightarrow \ldots$$

of higher derived limits.

Lemma 6.1.
(i) $\lim^i \Phi_s = 0$ *for* $i \geq n - s + 1$.
(ii) $\lim^i \Phi_{\leq s} \longrightarrow \lim^i \Phi$ *is an isomorphism for* $s \geq n - i + 1$.
(iii) If $\Phi(\beta) \cong \Phi(\alpha)$ *for* $\alpha \subset \beta$ *and* $|\beta| \leq n - i + 1$ *then* $\lim^i \Phi = 0$.

Proof. For each s the functor $\Phi_s \cong \prod_{\alpha \in K, |\alpha| = s} \Phi_\alpha$ is a product of atomic functors, i.e. $\Phi_\alpha(\beta) = 0$ if $\beta \neq \alpha$. In [NR2] the higher limits of atomic functors are calculated. We have $\lim^i \Phi_\alpha = \widetilde{H}^{i-1}(\ell_K(\alpha); \Phi(\alpha))$. Here,

$$\ell_K(\alpha) := \{\beta \in K : \alpha \cap \beta = \emptyset, \alpha \cup \beta \in K\}$$

denotes the link of the face α. In particular, $\dim(\ell_K)\alpha) \leq n - |\alpha| - 1$. Hence, these groups vanish for $i \geq n - |\alpha| + 1$, which proves the first part.

Since $\lim^j \Phi_{s+1} = 0$ for $j \geq n - s$, the second part follows from the above long exact sequences for higher derived limits.

Let $M := \Phi(\emptyset)$ and let $\text{cst}_M : \text{CAT}(K)^{op} \longrightarrow \text{AB}$ denote the constant functor. Then, $\lim^i \text{cst}_M = 0$ for $i \geq 1$ since $\text{CAT}(K)^{op}$ has terminal object and is contractible [BK]. By part (i) and (ii), we get

$$\begin{aligned} 0 = \lim{}^i \text{cst}_M &\cong \lim{}^i (\text{cst}_M)_{\leq n-i+1} \\ &\cong \lim{}^i \Phi_{\leq n-i+1} \cong \lim{}^i \Phi \end{aligned}$$

□

Proof of Proposition 5.1: We first consider the functors Π_i^U for $i \geq 3$. We want to calculate $\lim^j \Pi_i^U$ for $j \geq i$. If $i = 2k+1$ and $|\alpha| \leq n-i+1 = n-2k \leq n-k-1$, then $\Pi_{2k+1}^U(\alpha) = 0$ (Lemma 5.3) and as well as $\lim^j \Pi_i^U = 0$ for $j \geq i$ Lemma 6.1).

If $i = 2k$ and $|\alpha| \leq n - 2k \leq n - k - 1$, then $\Pi_i^U(\alpha) = \mathbb{Z}_p^\wedge$ and $\Pi_i^U(\alpha) \cong \Pi_i^U(\beta)$ for $\beta \subset \alpha$ by Lemma 5.3. Hence, by Lemma 6.1, we have again $\lim^j \Pi_i^U = 0$ for $j \geq i$.

The same argument shows that $\lim^j \hat{\Pi}_2^U = 0$ for $j \geq 2$.

Finally, we consider the functor Π_2^U which fits into the exact sequence

$$0 \longrightarrow \hat{\Pi}_2^U \longrightarrow \Pi_2^U \longrightarrow \Psi_2 \cong H^2(BT^K; \mathbb{Z}_p^\wedge) \longrightarrow 0.$$

For $j \geq 2$, we have $\lim^j H^2(BT^K; \mathbb{Z}_p^\wedge) = 0$ [NR1] as well as $\lim^j \hat{\Pi}_2^U = 0$. Hence, the same holds for Π_2^U, which finishes the proof in the complex case.

Now we consider the real case. By the same argument as for Π_{2k+1}^U, we have $\lim^j \Pi_1^{SO} = 0$.

If $j \geq i \geq 3$ we have $n - j + 1 \leq n - i + 1 \leq n - i/2 - 1/2$. This shows that $\Pi_i^{SO}(\beta) \cong \Pi_i^{SO}(\alpha)$ for $\alpha \subset \beta$ and $|\beta| \leq n - j + 1$ (Lemma 5.3) and that $\lim^j \Pi_i^{SO} = 0$ (Lemma 6.1).

For $i = 2$ we first consider the functor $\hat{\Pi}_2^{SO}$. If p is odd, then $\hat{\Pi}_2^{SO}(\alpha) \neq 0$ if and only if $|\alpha| = r - 1$. Hence, if $r \geq n + 2$, the functor $\hat{\Pi}^{SO}$ is trivial. For $r = n, n + 1$, $\hat{\Pi}_2^{SO}$ is a product of atomic functors only nontrivial either on faces of order n or $n - 1$. But in both case, we have $\lim^j \hat{\Pi}_2^{SO} = 0$ for $j \geq 2$.

If $p = 2$, then

$$\hat{\Pi}_2^{SO}(\alpha) \cong \pi_2(BSO(2(r - |\alpha|)) \cong \begin{cases} 0 & \text{for } r = |\alpha| \\ \mathbb{Z} & \text{for } r = |\alpha| + 1 \\ \mathbb{Z}/2 & \text{otherwise} \end{cases}.$$

If $r \geq n + 2$, then $\hat{\Pi}_2^{SO} = \mathrm{cst}_{\mathbb{Z}/2}$. If $r = n$, then we have two exceptional cases, namely $|\alpha| = n, n - 1$. Comparing $\hat{\Pi}_2^{SO}$ with $\mathrm{cst}_{\mathbb{Z}/2}$, we get an exact sequence

$$0 \longrightarrow \phi \longrightarrow \hat{\Pi}_2^{SO} \longrightarrow \mathrm{cst}_{\mathbb{Z}/2} \longrightarrow \nu \longrightarrow 0$$

with $\phi = \phi_{n-1}$ and $\nu = \nu_n$. And if $r = n$, then we have the same exact sequence, but $\phi = \phi_n$ and $\nu = 0$. In both cases $\lim^j \nu = 0$ for $j \geq 1$ and $\lim^j \phi = 0$ or $j \geq 2$ (Lemma 6.1). Splitting the long exact sequence into two short exact sequences, the long exact sequence for higher derived limits shows that

$$\lim{}^j \hat{\Pi}_2^{SO} = \lim{}^j \ \mathrm{cst}_{\mathbb{Z}/2} = 0$$

for $j \geq 2$.

Now, the same argument as for Π_2^U shows that $\lim^j \Pi_2^{SO} = 0$, which finishes the proof. □

7. Proof of Theorem 1.2

We start directly with proof. We continue to denote by $G(r)$ either $U(r)$ or $SO(2r)$ and by $T^r \subset G(r)$ the standard maximal torus of $G(r)$. Let $\hat{\xi}_r : DJ(K) \longrightarrow BT^r$ be a lift of $\xi_r : DJ(K) \longrightarrow BG(r)$. By construction of ξ_r, the restriction $\xi_r|_{BT^\alpha}$ can be lifted to map a $\xi_r^\alpha : BT^\alpha \longrightarrow BT^r$.

And this map as well as the underlying homomorphism $\iota^\alpha : T^\alpha \longrightarrow T^r$ is given by a coordinate wise inclusion. The restriction $\hat{\xi}_r|_{BT^\alpha} : BT^\alpha \longrightarrow BT^r$ is induced by a uniquely determined homomorphism $\hat{\iota}^\alpha : T^\alpha \longrightarrow T^r$. Both homomorphisms $\hat{\iota}^\alpha$ and ι^α are conjugate in $G(r)$ [N]. Since T^α is a topological cyclic group, they differ only by an element of the Weyl group $W_{G(r)}$. That is there exists $w \in W_{G(r)}$ such that $\hat{\iota}^\alpha = w\iota^\alpha$. If $G(r) = U(r)$, the Weyl group $W_{G(r)} = \Sigma_r$ acts via permutations on T^r and $w\iota^\alpha$ is again given by a coordinate wise inclusion. In particular there is a uniquely determined underlying set theoretic injection $g_\alpha : \alpha \longrightarrow [r] := \{1, ..., r\}$ which describes the coordinate wise inclusion $\hat{\iota}^\alpha$.

If $G(r) = SO(2r)$, then $W_{G(r)} \cong (\mathbb{Z}/2)^{r-1} \rtimes \Sigma_r$ where Σ_r again acts via permutations and where $(\mathbb{Z}/2)^{r-1}$ acts via complex conjugation on coordinates. In this case the underlying set theoretic injection $g_\alpha : \alpha \longrightarrow [r]$ describes $\hat{\iota}^\alpha$ only up to coordinate wise conjugation.

In both cases, if $\alpha \subset \beta$, the uniqueness of g_α says that $g_\beta|_\alpha = g_\alpha$. Combining all these maps defines then a map $g_{[m]} : [m] \longrightarrow [r]$ which extends to a non degenerate simplicial map $g : K \longrightarrow \Delta(r)$. That is to say that K has a r-coloring and shows that both, Part (ii) as well as Part (iii) of Theorem 1.2, imply the first part.

Now we start with an r-coloring $g : K \longrightarrow \Delta(r)$ and want to construct a lifting $\hat{\xi}_r : DJ(K) \longrightarrow BT^r$ of ξ_r. For each face α of K, the map g induces a set theoretic injection $g_\alpha : \alpha \longrightarrow [r]$. And g_α defines a coordinate wise inclusion $\eta^\alpha : BT^\alpha \longrightarrow BT^r$. By construction, for $\alpha \subset \beta$ we have $\eta^\beta|_{BT^\alpha} = \eta^\alpha$. All these maps fit together to define a map

$$\hat{\eta} : DJ(K) \simeq \mathrm{colim}_{\mathrm{CAT}(K)}\, BT^K \longrightarrow BT^r.$$

Let η denote the composition $DJ(K) \xrightarrow{\hat{\eta}} BT^r \longrightarrow BG(r)$. The restrictions $\eta|_{BT^\alpha}$ and $\xi_r|_{BT^\alpha}$ have both lifts to BT^r given by coordinate wise inclusion. These lifts differ only by an element of the Weyl group and $\eta|_{BT^\alpha}$ and $\xi_r|_{BT^\alpha}$ are homotopic. By Remark 5.2, this implies that η and ξ_r are homotopic and that ξ_r allows a lift to BT^r. This shows that Part (i) of Theorem 1.2 implies the other two parts and finishes the proof.

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