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University *of* Leicester

JANUARY EXAMINATIONS 2006

Subject MATHEMATICS

Title of paper MA2102 — LINEAR ALGEBRA

Time allowed <u>One and a half hours</u>

Instructions to candidates

This paper contains four questions.

Any number of questions may be attempted, but only the best *three* answers will be taken into account. Full marks may be obtained for answers to *three* questions.

All questions carry equal weight.

The Casio FX82, Casio FX83 or Casio FX85 calculator may be used in this examination; no other calculator is allowed.

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1. (a) **(4 marks)** Let $(V, <,>)$ be an inner product space over $\mathbb{R}$. Let $U, W \subset V$ be two subspaces of V. Explain what is meant by the orthogonal complement $U^\perp$ of U and what is meant by saying that V is the direct sum of U and W.

   (b) **(6 marks)** Let $<,>: \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ denote the standard inner product on $\mathbb{R}^3$, i.e. $< \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} > := xa + yb + zc$. Let $U \subset \mathbb{R}^3$ denote the subspace generated by the vector $u := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Calculate $U^\perp$ and find a basis for $U^\perp$.

   (c) **(6 marks)** Let V be a finite dimensional vector space over the field $\mathbb{F}$ and let $U, W \subset V$ be two subspaces such that $V = U \oplus W$. Let B_U be a basis for U and B_W a basis for W. Show that $B_U \cup B_W$ is a basis for V. If $\dim V = n$ and $\dim U = r$ what is $\dim W$?

   (d) **(4 marks)** Let $(V, <,>)$ be an inner product space over the complex numbers $\mathbb{C}$ and let $T : V \longrightarrow V$ be a linear map. Show that, if T is self adjoint, i.e. $< T(v), w > = < v, T(w >$, then $< T(v), v > \in \mathbb{R}$ for all $v \in V$.

   Total 20 marks

2. (a) **(4 marks)** Let V and W be vector spaces over the field $\mathbb{F}$. Explain what it means to say that the map $T : V \longrightarrow W$ is a linear transformation.

(b) **(6 marks)** Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the map defined by $T\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 2x - y \\ -x + 2y \\ x + y \end{pmatrix}$. Find the matrix A of T w.r.t. the bases $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ for $\mathbb{R}^2$ and $\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$ for $\mathbb{R}^3$. Calculate the dimension of the kernel and the dimension of the image of T.

(c) **(5 marks)** Let V be a finite dimensional vector space over the field $\mathbb{F}$, let B be a basis for V and let $T : V \longrightarrow V$ be a linear map. Show that T is an isomorphism if and only if the matrix A of T w.r.t. the basis B is invertible.

(d) **(5 marks)** Let V be a finite dimensional vector space over the field $\mathbb{F}$ and let $T : V \longrightarrow V$ be a linear map. Show that if $V = \ker T + \operatorname{im} T$, then $V = \ker T \oplus \operatorname{im} T$.

Total 20 marks

3. (a) **(4 marks)** Let V be vector space over the field $\mathbb{F}$ and let $T : V \longrightarrow V$ be a linear map. Explain what is an eigenvalue and an eigenvector for T.

(b) **(6 marks)** Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear map defined by $T\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 5x + y \\ -x + 3y \end{pmatrix}$. Calculate the eigenvalues for T and an eigenvector for each eigenvalue. Does there exist a basis for $\mathbb{R}^2$ consisting of eigenvectors of T ?

(c) **(5 marks)** Let V be a finite dimensional vector space over the field $\mathbb{F}$, let B be a basis for V and let $T : V \longrightarrow V$ be a linear map. Let A be the matrix for T w.r.t. the basis B. Show that if there exists a basis for V consisting of eigenvectors of T, then the matrix A is diagonalizable.

(d) **(5 marks)** Let V be a finite dimensional vector space over $\mathbb{R}$ and let $T : V \longrightarrow V$ be a linear map. For $\lambda \in \mathbb{R}$ we define $E_\lambda := \{v \in V : T(v) = \lambda v\}$. Show that, if $T^2 = T$, i.e. $T(T(v)) = T(v)$ for all $v \in V$, then $V = E_1 \oplus E_0$. (Hint: $v = T(v) + v - T(v)$.)

Total 20 marks

4. Which of the following claims are true and which are false ? Justify your answer.

(a) **(2 marks)** The map $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $T(x) := x + 1$ is linear.

(b) **(3 marks)** There exists a linear map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $\operatorname{im} T = \mathbb{R}^3$.

(c) **(3 marks)** There exists a linear map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $\dim(\ker T) = \dim(\operatorname{im} T) = 1$.

(d) **(3 marks)** Let V be a finite dimensional vector space over the field $\mathbb{F}$ and let $U \subset V$ be a subspace. Then there exists a subspace $W \subset V$ such that $V = U \oplus W$.

(e) **(3 marks)** For every linear map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ there exists a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.

(f) **(3 marks)** The map $s : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $s(x, y) := 3xy$ defines an inner product on $\mathbb{R}$.

(g) **(3 marks)** Let $(V, <,>)$ be an inner product space such that V is finite dimensional and $\dim V \geq 1$. Then there exists a subspace $U \subset V$ of V such that $U = U^\perp$.

Total 20 marks