MA2102 Linear Algebra

Spanning Sets, Linear Independece and Basis

Spanning sets

<u>Definition:</u>
Let V be a vector space over $\mathbb{F}$. And let
$S := \{v_1, ..., v_r\} \subset V$ be set of vectors in V.
A vector $v \in V$ is a linear combination of the elemnts of S, if there exists $\lambda_1, ...\lambda_r \in \mathbb{F}$ such that

$$v = \sum_{i=1}^{r} \lambda_i v_i$$

The linear span or span of S, denoted by $\mathrm{sp}(S)$ is the set of all linear combinations of elements of S.

$$S := \{v \in V | v] \sum_{i=1}^{r} \lambda_i v_i, \lambda_i \in \mathbb{F}\}$$

<u>Remark:</u>
$\mathrm{sp}(S) \subset V$ is a subspace.

Linear Independence

<u>Definition:</u>
Let V be a vector space over $\mathbb{F}$. And let
$S := \{v_1, ..., v_r\} \subset V$ be set of vectors in V.
The set S is a called linearly independent if

$$\sum_{i=1}^{r} \alpha_i v_i = 0 \Longrightarrow \alpha_i = 0 \text{ for all } i = 1, ..., r.$$

i.e. the only way to represent 0 as a linear combination of elemnts of S is to take zero multiples of the vectors.

If S is not linearly independent, then we call S linearly dependent, i.e. there exist $\alpha_1, ..., \alpha_r \in \mathbb{F}$, not all equal to 0, such that
$0 = \sum_{i=1}^{r} \alpha_i v_i$.

Remark:

- $S = \{v_1, ..., v_r\} \subset V$ is linearly independent iff for all $v \in \text{sp}(S)$ then there exists exacltly one way to write v as a linear combination
$v = \sum_{i=1}^{r} \alpha_i v_i$.

- $S = \{v_1, ..., v_r\} \subset V$ is linearly dependent iff there exists $v_j \in S$ such that $v_j = \sum_{i \neq j} \beta_i v_i$ for some $\beta_i \in \mathbb{F}$.

Basis

Definition:

Let V be a vector space over $\mathbb{F}$. A set $B := \{v_1, ..., v_n z\} \subset V$ is called a basis for V, if the following holds:
(i) B is linearly independent
(ii) B is a spanning set for V, i.e. $\text{sp}(B) = V$.

Remark

If a set $B = \{v_1, ..., v_n\}$ is a basis for V, then each vector of V can be written in a unique way as a linear combination of the lements of B.