MA2102 Linear Algebra 06/07

Question Sheet 4

Group work

This question sheet is again group work. The groups are as for Question sheet 2 and as follows:

Group 1: T Evans, A James, SWR Lowry, ED Mander
Group 2: S Coombes, J Power, H Ragunathan, CR Tomasello
Group 3: A Ahmed, S Ahmed, PF Codjoe, F Saleh
Group 4: SN Burford, NE Gingell, JC Stuckey, AS Thornton
Group 5: M Afxentiou, P Demetriadou, E Eleftheriou, A Pagdati
Group 6: D Ganatra, A Greaves, SG Ratnam, KTF van der Meeren
Group 7: HS Jandu, J Rowley, R Shah, K Tedds
Group 8: I Cadwallader, DM Page-Winn, B Palan, T Wilkinson
Group 9: Z Carter, S Ibrahim, CPT Le, OY Omer
Group 10: H Goodman, L Li, Z Xi, H Xu
Group 11: FA Hefford, M Hughes, JD Parsons, JM Stimpson
Group 12: SK Birdie, AK Boolaky, D Li, J Li,

You are only allowed to submit one set of solutions for each group. Write only all the names (family name) of those of your group on the sheets which contributed to the homework and staple it.

If necessary, you can contact other people of your group by e-mail. The home page of the University provides a link to an e-mail directory with a search function.

This question sheet will be assessed and each group is required to hand in solutions for the questions listed in the work set at the 10:30 lecture on **Thursday, 7 December**. Solutions will be discussed in the problem class at 10:30 on Wednesday, 13 December.

Work set: 1, 2, 3(i), 3(ii). 4, 8 and 9(i).

1. **(3 marks)** Let $<,>: \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}$ denote the standard inner product for the complex vector space $\mathbb{C}^3$. Let $v := \begin{pmatrix} 2 \\ 1+i \\ i \end{pmatrix}$ and $w := \begin{pmatrix} 2-i \\ 2 \\ 1+2i \end{pmatrix}$. Calculate $< v, w >$, $< w, v >$, $\|v\|$, $\|w\|$ and $\|v+w\|$.

2. **(4 marks)** Apply the Gram-Schmidt process to the basis $\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}\}$ of $\mathbb{R}^3$ and construct an orthonormal basis for $\mathbb{R}^3$. The vector space $\mathbb{R}^3$ is equipped with the standard dot product.

3. Let $s : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a map defined by $s(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}) := 2xa + xb + ya + 2yb$.

   (i) **(4 marks)** Show that s defines an inner product on $\mathbb{R}^2$.

   (ii) **(2 marks)** Let $v := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Can you find a vector $w \in \mathbb{R}^2$, which is perpendicular to v w.r.t. the inner product defined in the first part ?

4. **(4 marks)** Let V be an vector space over $\mathbb{R}$ with inner product $< \, , \, >: V \times V \longrightarrow \mathbb{R}$. Let $v, w \in V$ be two vectors. Show that $v = w$ if and only if $< u, v > = < u, w >$ for all $u \in V$.

5. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear map defined by $T(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) := \begin{pmatrix} 3x + 2y - 2z \\ -4x - 3y + 2z \\ -z \end{pmatrix}$.

   (i) Calculate all eigenvalues of T and for each eigenvalue an eigenvector.
   (ii) Does there exists a basis for $\mathbb{R}^3$ which consists of eigenvectors of T ?

6. (i) Show that every square matrix $A \in \mathbb{C}^{n \times n}$ has an eigenvalue.
   (ii) Let$A, B \in \mathbb{C}^{n \times n}$ be two square matrices, $\mathbb{C}$ denotes the complex numbers. Show that if B is invertible, then there exists $z \in \mathbb{C}$ such that $A + zB$ is not invertible. (Hint: Examine $\det(A + zB)$ and think of z as a variable.)
   (ii) Find two matrices $A, B \in \mathbb{C}^{2 \times 2}$ such that $A + tB$ is invertible for all $t \in \mathbb{R}$.

7. Let $C[0, 1]$ denote the vector space of all continuous functions $f : [0, 1] \longrightarrow \mathbb{R}$. Show that $< f, g > := \int_0^1 f(x)g(x)dx$ defines an inner product on $C[0, 1]$.

8. **(4 marks)** Let $(V, <, >)$ be a finite dimensional inner product space and $T : V \longrightarrow V$ be a linear map. Show that, if $\|T(v)\| = \|v\|$ for all $v \in V$, then T is an isomorphism. Is it also true that, if T is an isomorphism, then $\|T(v)\| = \|v\|$ for all $v \in V$?

9. Let $(V, <, >_V)$ and $W, <, >_W$ be two inner product spaces over $\mathbb{R}$ and let $T : V \longrightarrow W$ be a linear map. Let $B_V := \{v_1, ..., v_n\} \subset V$ and $B_W := \{w_1, ..., w_r\} \subset W$ be orthonormal bases for V respectively W.

   (i) **(4 marks)** Show that for every $w \in W$ there exists a uniquely determined $v \in V$ such that $< v, u >_V = < w, T(u) >_W$ for all $u \in U$. (For the uniqueness you may want to use Question 4 from this question sheet. For the existence, write $v = \sum_{k=1}^n \lambda_k v_k$ and try to determine the coefficients λ_k.)

   (ii) Show that there exists a well defined linear map $S : W \longrightarrow V$ such that $< S(w), v >_V = < w, T(v) >_W$ for all $v \in V$ and $w \in W$.
   (The map $S : W \longrightarrow V$ is a called the adjoint of $T : V \longrightarrow W$. And if $V = W$ and $S = T$ then T is called self adjoint.)

   (iii) Let $A = (a_{ij})$ be the matrix for T w.r.t. B_V and B_W. Show that $a_{ij} = < w_j, T(v_i) >_V$.

   (iv) Let A' denote the matrix for S w.r.t. the bases B_W and B_V. Show that $A' = A^T$, i.e. A' is the transpose of A.

10. Which of the following matrices are diagonalizable ?
    a) $A := \begin{pmatrix} \pi & e \\ e & \pi \end{pmatrix}$ b) $B := \begin{pmatrix} i & 1 + i \\ 0 & i \end{pmatrix}$ c) $C := \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$