Lemma: (Exchange Lemma)
Let V be a finite dimensional vector space over the field $\mathbb{F}$. Let $S := \{v_1, ..., v_n\}$ be a spanning set for V and $T := \{u_1, ..., u_r\} \subset V$ a subset set of linearly independent vectors. Then, $r \leqslant n$ and we can exchange r of vectors of S for the set T with out changing the span, i.e. there exists vectors $v_{i_{r+1}}, ..., v_{i_n}$ such that $V = \text{span}(u_1, ..., u_r, v_{i_{r+1}}, ..., v_{i_n})$.

Consequences:

Theorem
Let V be a finite dimensional vector space over $\mathbb{F}$, and let $S = \{v_1, ..., v_n\}$ and $T = \{w_1, ..., w_r\}$ be two subsets of vectors of V. If $V = \text{span}(S)$ and T is linearly independent, then $r \leqslant n$.

Definition
Let V be a finite dimensional vector space over $\mathbb{F}$. Then, $\dim V$ is the number of vectors in any basis; i.e. if $B := \{v_1, ..., v_n\}$ is a basis for V, then $\dim V = n$.

Theorem
Let V be a finite dimensional vector space over $\mathbb{F}$ and $U \subset V$ be a subspace. Then, $\dim U \leqslant \dim V$ and every basis of U can be extended to a basis of V.