Geometric construction of m-cluster categories

Karin Baur
University of Leicester

Joint work with Robert Marsh

Type A_n: Transactions (math.RT/0607151)
Type D_n: IMRN (math.RT/0610512)

Cluster algebras

Introduced by Fomin/Zelevinsky

Subrings of $\mathbb{Q}(u_1, \ldots, u_m)$.

Defined via generators, the **cluster variables** (constructed recursively).

Clusters: subsets of fixed cardinality.

Motivation: algebraic framework for

- total positivity
- canonical basis of quantum groups (Lusztig/Kashiwara).

Laurent phenomenon: Cluster variables are in $\mathbb{Z}[u_1^{\pm}, \ldots, u_m^{\pm}]$, hence the cluster algebra is in $\mathbb{Z}[u_1^{\pm}, \ldots, u_m^{\pm}]$

Connections to

- Poisson geometry, Teichmüller spaces
- Grassmannians
- Y-systems

Cluster categories

Introduced by BMRRT, CCS.

Q quiver, underlying graph: ADE.

$D^b(kQ)$: bounded derived category of fin.dim. kQ-modules ($k = \overline{k}$).

Cluster category, $\mathcal{C}$: orbit category of $D^b(kQ)$ under canonical automorphism. Independent of orientation of Q.

$\mathcal{C} := D^b(kQ)/\tau^{-1} \circ [1]$

Correspondence:

Gabriel:	indec. kQ-mods	$\longleftrightarrow$	pos. roots
cluster case :	indec. obj. of $\mathcal{C}$	$\longleftrightarrow$	almost pos. roots
		$\longleftrightarrow$	cluster variables
	tilting objects of $\mathcal{C}$	$\longleftrightarrow$	clusters

(BMRRT '06 types A, D, E; CCS '06 type A).

m-cluster category, $\mathcal{C}^m$: (Keller, '05)

$\mathcal{C}^m := D^b(kQ)/\tau^{-1} \circ [m]$.

$\mathcal{C}^m$ is triangulated (Keller), Krull-Schmidt (BMRRT). Calabi-Yau of dimension $m+1$ (Keller).
Studied by Keller, Thomas, Wralsen, Zhu, B-Marsh, Assem-Brüstle-Schiffler-Todorov.

Goal: Describe $\mathcal{C}^m$ using diagonals of a polygon (type A_n) and arcs in a punctured polygon (type D_n).

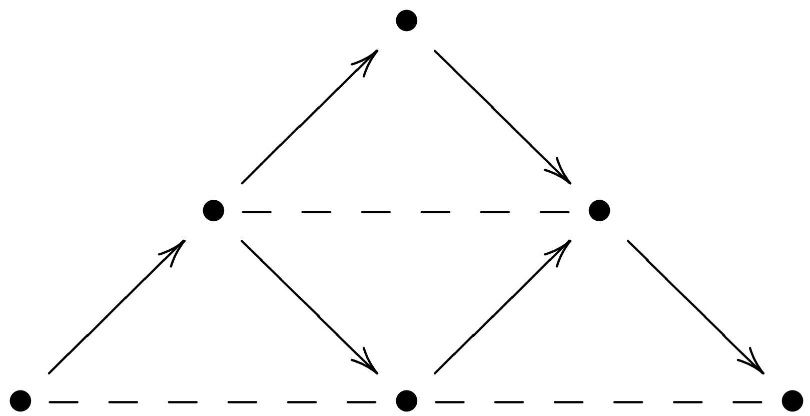
Translation Quiver

Definition: A **translation quiver** is a pair (Γ, τ) where

- $\Gamma = (\Gamma_0, \Gamma_1)$ is a locally finite quiver without loops
- $\tau : \Gamma_0' \to \Gamma_0$ is an injective map, $\Gamma_0' \subset \Gamma_0$
- $\forall\ x \in \Gamma_0,\ \ y \in \Gamma_0'$:
  $\#\ \{\text{arrows } x \to y\} = \#\ \{\text{arrows } \tau y \to x\}$

τ is the **translation** of (Γ, τ)

Example:



τ: indicated by $--$, maps from right to left.

Vertices in $\Gamma_0 \setminus \Gamma_0'$ are called **projective**.

More general example:

A a finite dimensional algebra over a field $k = \overline{k}$.

Quiver: Γ_0: isom. classes of indecomposable modules in mod A

Γ_1: irreducible maps

Auslander-Reiten quiver of A

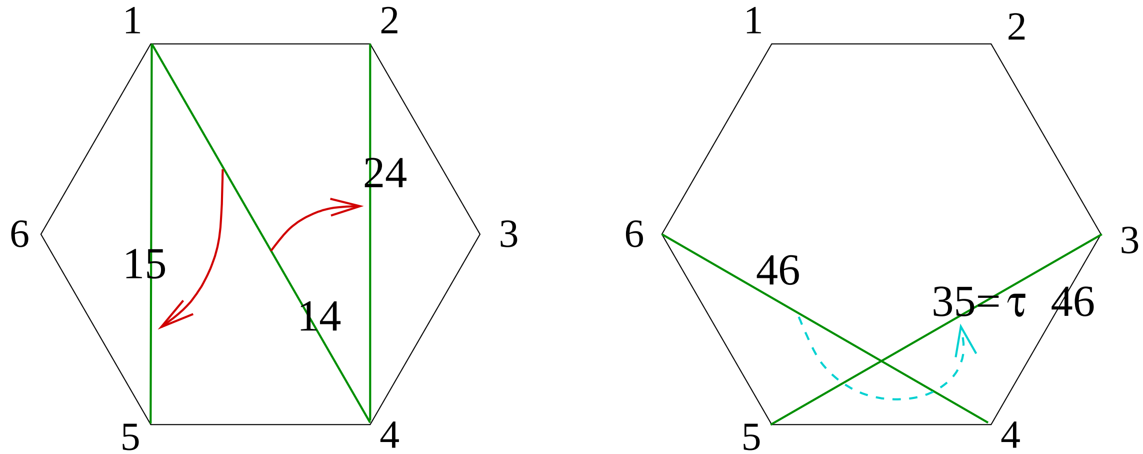
Map τ: "Auslander-Reiten translation"

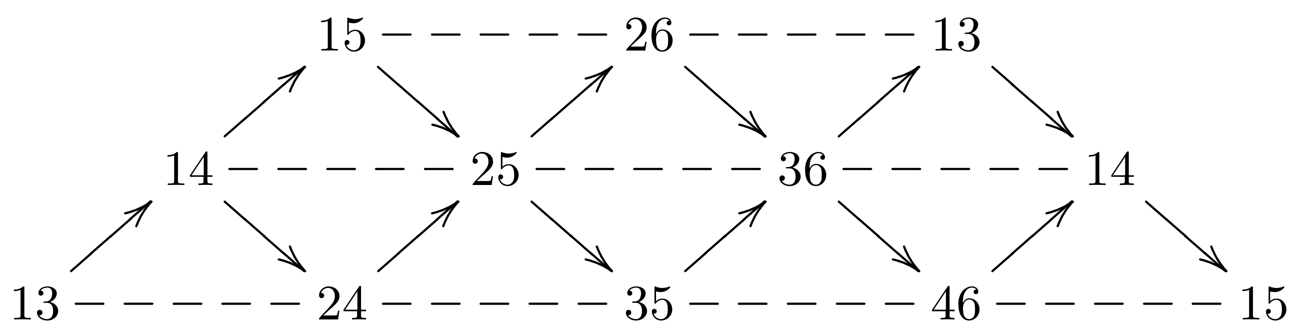
Example Hexagon:

Γ_0**: diagonals** (ij)

Γ_1**: arrows** $(ij) \to (i, j+1)$**,** $(ij) \to (i+1, j)$,
provided the image is a diagonal $(i, j \in \mathbb{Z}_6)$.

Translation τ**:** $(ij) \to (i-1, j-1)$
(anti-clockwise rotation about center, 60°)





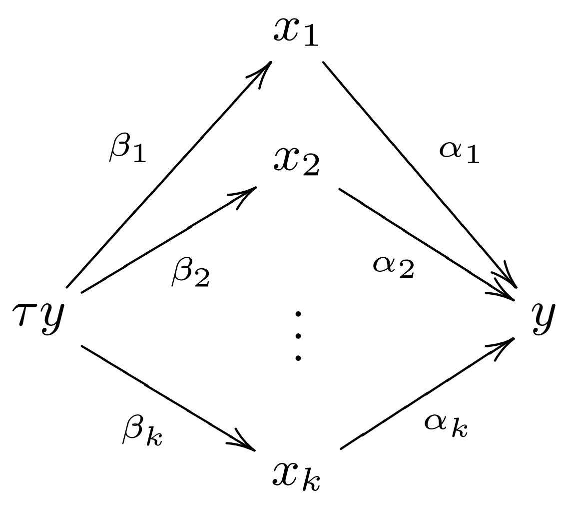
(Γ, τ) is called **stable** if $\Gamma_0' = \Gamma_0$.

Mesh category

$\Gamma = (\Gamma, \tau)$ a translation quiver, $y \in \Gamma_0$.

Let $\alpha_i : x_i \to y$ be the arrows to y $(i = 1, \dots, k)$ and $\beta_i : \tau y \to x_i$ the corresp. arrows from τy.

The **mesh ending at y** is the subquiver



Let $k\langle\Gamma\rangle$ be the free k-linear category on Γ.

Objects	$\longleftrightarrow$	Γ_0
morphisms	$\longleftrightarrow$	paths, under composition

Extend this k-linearly.

Definition

The **mesh category of Γ** is

$$k\langle\Gamma\rangle/\mathcal{R}$$

where $\mathcal{R}$ is given by the $m_y := \sum_{i=1}^{k} \beta_i\alpha_i$, over all meshes as above (i.e. for all $y \in \Gamma'_0$)

Tzanaki complex

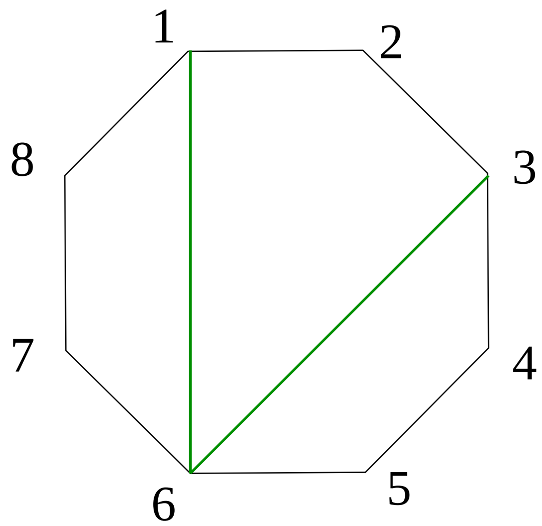
For $n, m \in \mathbb{N}$ let Π be an $nm+2$-gon, label the vertices $1, 2, \ldots, nm+2$.

An **m-diagonal** is a diagonal (ij) dividing Π into an $mj+2$-gon and an $m(n-j)+2$-gon $(1 \leq j \leq \frac{n-1}{2})$.

Obtain a simplicial complex on the set of m-diagonals.
Simplices: collections of non-crossing m-diagonals.
Maximal simplex: contains $n-1$ elements.

Example Octagon: Here $n = 3, m = 2$:

$nm + 2 = 8,\ jm + 2 = 4, 6$



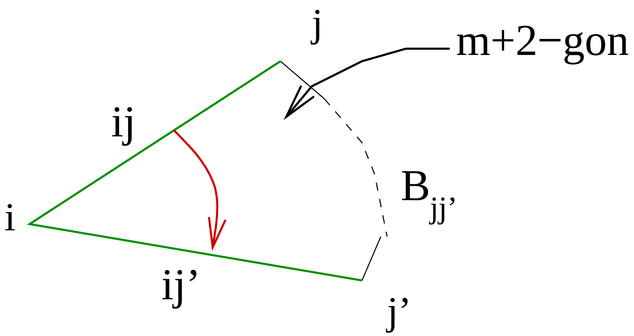
A maximal simplex: $\{(16), (36)\}$

Quiver $\Gamma(n,m)$, type A_{n-1}.

We define a quiver $\Gamma(m,n) = (\Gamma, \tau_m)$ as follows:

Γ_0**:** m**-diagonals**

Γ_1**:** $(ij) \to (ij')$ if (ij), $B_{jj'}$ and (ij') span an $m+2$-gon ($B_{jj'}$ is boundary j to j').



τ_m: rotation anti-clockwise (about center), angle $m\frac{2\pi}{nm+2}$.

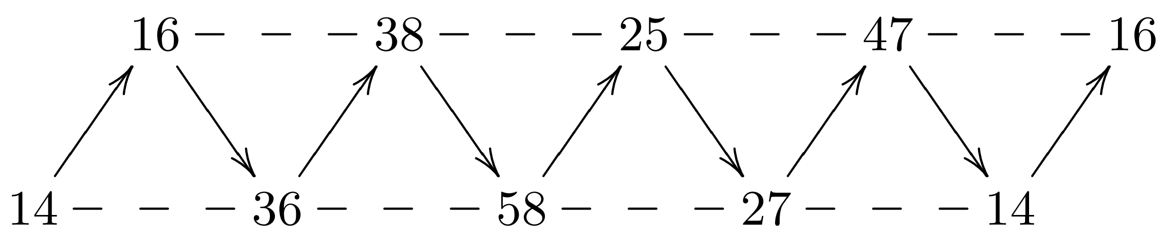
If $m = 1$: usual diagonals.

Proposition:

$\Gamma(n, m)$ is a translation quiver.

Let $\mathcal{C}(n, m)$ be the mesh category of $\Gamma(m, n)$.

E.g. for $n = 3, m = 2$:



Equivalence of categories

Q a Dynkin quiver of type A_{n-1} (type D_n)
$D^b(kQ)$ bounded derived category of fin. dim. kQ-modules
τ: Auslander-Reiten translate,
$F_m := \tau^{-1} \circ [m]$.

Theorem ($m = 1$: Caldero-Chapoton-Schiffler. $m \geq 1$: B.-Marsh).

$$\mathcal{C}(n, m) \;\cong\; \text{ind } D^b(kQ)/F_m$$

Proof uses Happels description of (AR-quiver of) $D^b(kQ)$ and combinatorial analysis of $\Gamma(n, m)$ (resp. of $\Gamma_{\odot}(n, m)$).

In above example: $D^b(A_2)/F_2$.

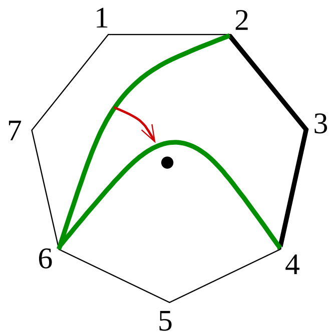
Quiver $\Gamma_{\odot}(n,m)$, type D_n.

We define a quiver $\Gamma_{\odot}(m,n) = (\Gamma, \tau_m)$:

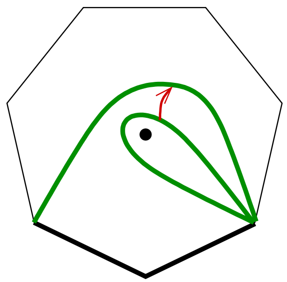
Γ_0: tagged m-arcs of a punctured $nm - m + 1$-gon

Γ_1: m-moves

E.g $(ij) \to (ik)$ if (ij), B_{jk} (boundary j to k) and (ik) span a (degenerate) $m+2$-gon.



move (i)


move (ii)

τ_m: rotation anti-clockwise (about center).

m-th power of translation quivers

(Γ, τ) a translation quiver.

A path $x_0 \to x_1 \to \cdots \to x_{m-1} \to x_m$ is **sectional** if $\tau x_{i+1} \neq x_{i-1}$ for $i = 1, \ldots, m-1$ (for which τx_{i+1} is defined).

Define Γ^m as the quiver with **vertices** Γ_0 and **arrows: sectional paths in Γ of length m**.

Let τ^m be $\tau \circ \tau \circ \cdots \circ \tau$ (m times).

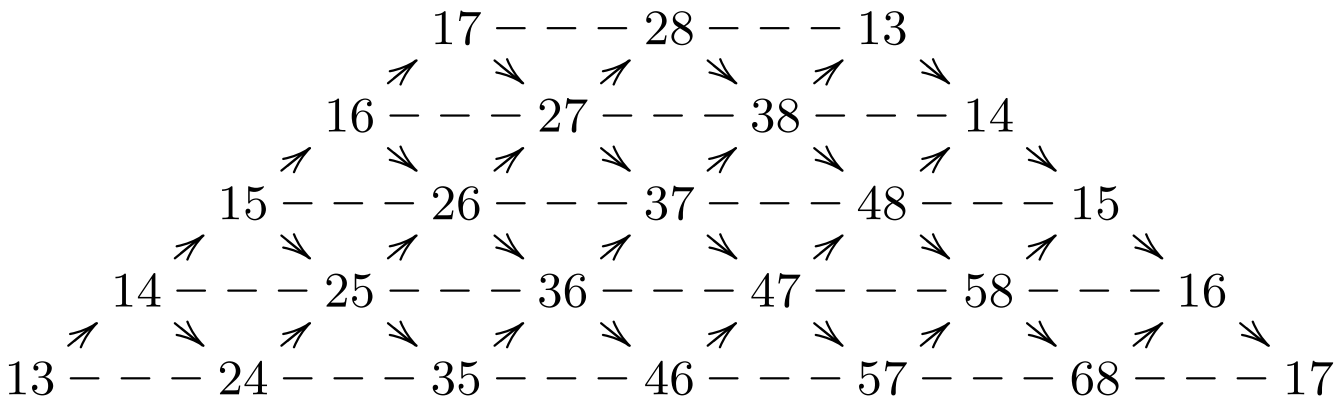
Theorem: Let (Γ, τ) be a translation quiver such that if y is projective and $x \to y$ then x is projective.
Then (Γ^m, τ^m) is a translation quiver.

Example: a stable (Γ, τ).

Theorem: (type A_{n-1})
ind $D^b(kQ)/F_m$ is a full subcategory of
$(\text{ind } D^b(kQ)/F_1)^m$

So: $\Gamma(n,m)$ is a full subquiver of
$(\Gamma(nm,1))^m = (\Gamma(\text{cluster category}))^m$

Example: Type A_5, gives $\mathcal{C}(3,2)$ from above.

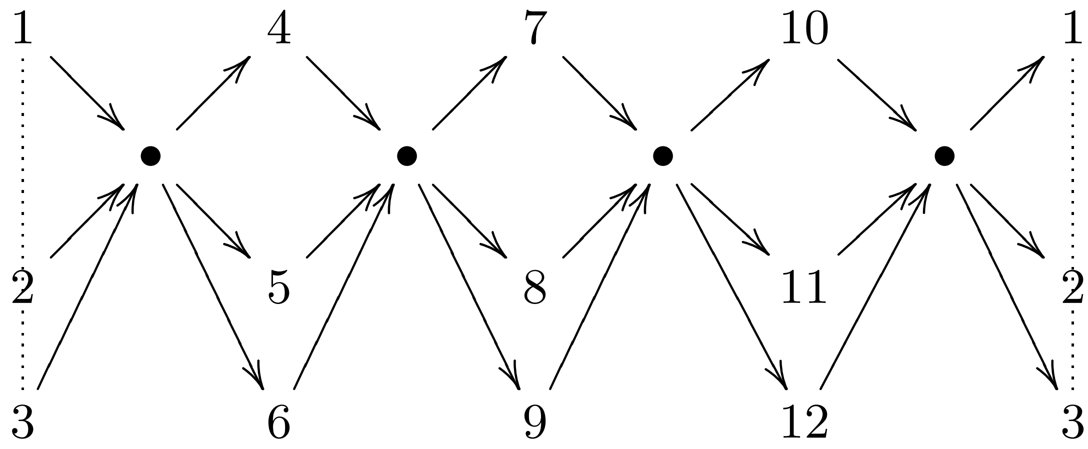


Obtain two other components:

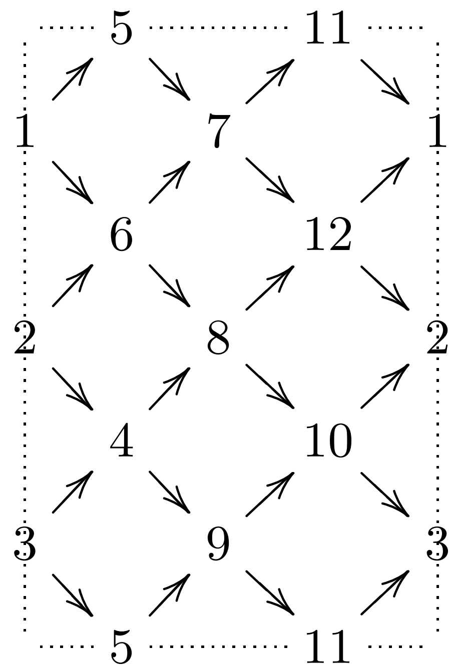
$D^b(A_3)/S$ twice.

Working on: other types, describing other components.

E.g. type D_4:



Second power is a torus:



Solution: restrict sectional paths of length m.

Theorem: (type D_{nm-m+1})
The restricted m-th power
$\mu_m(\Gamma(D_{nm-m+1}, 1), \tau^m)$ is the union of the
following connected components:

$$\mu_m(\Gamma(D_{nm-m+1}, 1), \tau^m) = \Gamma_{\odot}(n, m) \cup \bigcup_{k=1}^{m-1} \Gamma(D^b(A_{n-1})/\tau^{nm-m+1})$$

$(\Gamma(D^b(A_{n-1})/\tau^{nm-m+1})$ denotes the
Auslander-Reiten quiver of
$D^b(A_{n-1})/\tau^{nm-m+1})$.