MA1251 Chaos and Fractals: Problem Set No. 3

Due: Tuesday, 20 May 2003, 11:30am

1. (Devil's staircase) Suppose that we pick a point at random from the Cantor set. What's the probability that this point is smaller than x, where $0 \leq x \leq 1$ is some fixed number? The answer is given by a function $P(x)$ called the **devil's staircase**.

   (a) It is easiest to visualise $P(x)$ by building it up in stages. First consider the set S_0 from the lectures. Let $P_0(x)$ denote the probability that a randomly chosen point in S_0 is smaller than x. Show that $P_0(x) = x$.

   (b) Now consider S_1 and define $P_1(x)$ analogously. Draw the graph of $P_1(x)$. *Hint:* It should have a plateau in the middle.

   (c) Draw the graphs of $P_n(x)$, for $n = 2, 3, 4$. Be careful about the widths and heights of the plateaus.

   (d) The limiting function $P_\infty(x)$ is the devil's staircase. Is it continuous? What would a graph of its derivative look like?

2. (Snowflake) To construct the famous fractal known as the *von Koch snowflake curve*, use an equilateral triangle for S_0. Then do the von Koch procedure illustrated in the Figure on each of the three sides.

   (a) Draw S_2 and S_3.

   (b) The von Koch snowflake curve is the limiting curve $S = \lim_{n\to\infty} S_n$. Show that it has infinite arc length.

   (c) Find the area of the region enclosed by S.

   (d) Find the similarity dimension of S.

3. (Sierpinski carpet) Consider the process shown in the Figure. The closed unit box is divided into nine equal boxes, and the open central box is deleted. Then this process is repeated for each of the eight remaining sub-boxes, and so on.

   (a) Sketch the next stage S_3.

   (b) Find the similarity dimension of the limiting fractal, known as the *Sierpinski carpet.*

   (c) Show that the Sierpinski carpet has zero area.

4. (Your fractal) Design procedure for constructing an exactly self-similar fractal of your own. (Just like when discovering new stars, you are entitled to give it a name.) Find the similarity dimension of *Your fractal.*